Radiation and Refraction of Sound Waves Through a Two-Dimensional Shear Layer
Instability wave

An instability wave is admitted by LEE at the excitation frequency

\[ \alpha = 0.4145 - 0.0377i \ (1/m) \]

at \( \omega = 76 \ \text{rad/s} \Rightarrow \text{An instability (growing) wave is triggered!} \)
Analytical solution of the acoustic wave (Dahl, 1999)

The acoustic pressure

\[ p_a(k, y) = -\frac{iA^* M_j^2}{2\pi} \sqrt{\frac{\pi}{B'}} \int_{-\infty}^{\infty} e^{-k^2/4B'} \left[ \xi_1(k, y) \int_{y}^{\infty} \frac{\bar{\rho}(y_0)(\omega - \bar{u}(y_0)k)}{\Delta(k, y_0)} e^{-B' y_0^2} \xi_2(k, y_0) dy_0 \right. \\
\left. + \xi_2(k, y) \int_{0}^{y} \frac{\bar{\rho}(y_0)(\omega - \bar{u}(y_0)k)}{\Delta(k, y_0)} e^{-B' y_0^2} \xi_1(k, y_0) dy_0 \right] dk \]

\( \xi_1(k, y) \) \( \xi_2(k, y) \) : solutions to the characteristic equation of the LEE as two IVP's with each of the boundary conditions.

Characteristic Equation:

\[ \frac{d}{dy} \left( \frac{1}{\bar{\rho}(\omega - k\bar{u})^2} \frac{dP}{dy} \right) + \left( \frac{1}{\gamma \bar{\rho}} - \frac{k^2}{\bar{\rho}(\omega - k\bar{u})^2} \right) P = 0 \]

Boundary Conditions:

\[ \frac{dP}{dy} = 0, \ y = 0 \]
\[ \frac{dP}{dy} = i\beta P, \ y \to \infty \]

The Wronskian:

\[ \Delta(k, y) = \xi_1(k, y) \frac{\partial \xi_2(k, y)}{\partial y} - \xi_2(k, y) \frac{\partial \xi_1(k, y)}{\partial y} \]
Numerical scheme and boundary conditions

- **4th-order seven-point-stencil optimized upwind Dispersion-Relation-Preserving (DRP) scheme** (Zhuang & Chen, 2003)

- At the lower boundary: symmetry boundary condition
  - A ghost point is introduced right below each of the node at the lower boundary in implementing the condition $\frac{\partial p}{\partial y} = 0$ at $y = 0$. (Tam, 1996)

- At the left, right and upper boundaries: Perfectly-Matched-Layer (PML) B.C. in unsplit physical variables (Hu, 2001)
  - A characteristics-based boundary condition at the outer boundaries of the PML region
  - Works very well in the current highly non-uniform mean flow condition
Grid, time step and computational cost

Grid

- In the $x$-direction: uniform spacing $\Delta x = 1.5$ m.
- In the $y$-direction: grid clustered in the shear layer and the source region.
  - Grid density specified as a function of $y$
    \[
    \left( \frac{dy}{d\xi} \right)_j = 1 - 0.6 \exp[-(\ln 2)y_j^2 / 50]
    \]
    $\xi$: transformed coordinate of $y$.
  - $\Delta y$ ranges from 0.4 m in the shear layer to 1.0 m in the far field.

![Diagram: Grid, time step and computational cost](image)

- White area: the PML region
- Grey area: the interior
Grid, time step and computational cost

- Time step
  - \( \Delta t = 2.0668372 \times 10^{-5} \) s (or \( T/4000 \), \( T \): period of the acoustic source)

- Computational cost
  - 800,000 time steps (200 periods) before a periodic solution is reached
  - CPU time: \( \sim 7.5 \) hours
  - Memory usage: \( \sim 10 \) MB

- Computer configuration
  - CPU: one Athlon XP 1800+ at 1533 MHz
  - Physical memory: 512 MB DDR memory at 266 MHz
  - Operating system: Windows XP
  - Compiler: DIGITAL Visual Fortran, version 6.0.A
The total solution from the numerical solution

- Total solution = Acoustic wave + Instability wave
  - Excellent performance of the PML boundary condition

- Acoustic wave = Total solution - Instability wave
- How to construct the instability wave numerically?
Constructing the instability wave

- With a different source upstream of the domain of interest
  - Produces a stronger instability wave and a weaker acoustic wave in the domain of interest
  - A dipole source of the same excitation frequency is preferred.
    - The acoustic wave that it produces tends to be more compact around the source region.
- On the same grid
- Nearly “pure” instability wave in the domain of interest

- Amplitude and phase are adjusted based on the node at \( x = 147 \) and \( y = 0 \)
**Acoustic wave = Total solution - Instability wave**
Comparison with the analytical solution

Acoustic wave solution, \(-50 \leq x \leq 150, y = 50\)

\[ P \]

\[ x \]

Acoustic wave solution, \(-50 \leq x \leq 150, y = 15\)

\[ P \]

\[ x \]

Acoustic wave solution, \(x = 100, 5 \leq y \leq 50\)

\[ P \]

\[ y \]