Sensitivity Analysis of Broadband Shock-Associated Noise based on Navier-Stokes Equations Decomposition

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Motivation

- More than 107,000 sailors are living and working aboard US Navy Ships.
- Over a billion dollars were spent each year by US Department of Veterans Affairs for hearing loss disability benefits in the last decade\(^1\)

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Components of Jet Noise

- Two main components of jet noise - turbulent mixing noise and shock-associated noise.
- Broadband shock-associated noise is generated due to \textit{coherent interaction} of large-scale structures with the shock-cell structure.
- Mainly dominant in the sideline and upstream direction.

\begin{itemize}
  \item Fine Scale Noise
  \item Shock Associated Noise
  \item Large Structure Noise
  \item Potential Core
  \item Shear Layer
\end{itemize}

\begin{align*}
\text{SPL per unit St} &= 130 \\
\text{SPL}_{\text{max}} &= 134.07 \\
\text{SPL per unit St} &= 120 \\
\text{SPL}_{\text{max}} &= 129.58 \\
\text{SPL per unit St} &= 110 \\
\text{SPL}_{\text{max}} &= 129.51 \\
\text{SPL per unit St} &= 100 \\
\text{SPL}_{\text{max}} &= 130.25 \\
\text{SPL per unit St} &= 90 \\
\text{SPL}_{\text{max}} &= 130.53 \\
\text{SPL per unit St} &= 80 \\
\text{SPL}_{\text{max}} &= 131.54 \\
\text{SPL per unit St} &= 70 \\
\text{SPL}_{\text{max}} &= 129.44 \\
\text{SPL per unit St} &= 60 \\
\text{SPL}_{\text{max}} &= 127.45 \\
\text{SPL per unit St} &= 50 \\
\text{SPL}_{\text{max}} &= 127.43 \\
\text{SPL per unit St} &= 40 \\
\text{SPL}_{\text{max}} &= 124.7 \\
\text{SPL per unit St} &= 30 \\
\text{SPL}_{\text{max}} &= 132.92
\end{align*}

\begin{figure}
\begin{center}
\includegraphics[width=\textwidth]{noise_spectra.png}
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\begin{figure}
\begin{center}
\includegraphics[width=\textwidth]{noise_components.png}
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\begin{footnotesize}
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Acoustic Analogy based on the Navier-Stokes Equations

- We start with the Navier-Stokes equations as the governing equations.
- The *field-variables* such as density, velocity, pressure, and temperature are decomposed into base flow, aerodynamic fluctuations, and acoustic fluctuations.
- The aerodynamic turbulent fluctuations (sources) are brought to the right-hand side.
- The radiating acoustic components (propagators) are brought on the left-hand side.
- The spectral density of the radiating components is obtained by *convolving* the vector Green’s function with the source terms present on the right-hand side.

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Decomposition of the Field-Variables

- Decomposition of field variables:

\[ q = \bar{q} + \dot{q} + \hat{q} + q' + q'' \]

- \( \bar{q} \) = Base quantity.
- \( \dot{q} \) = Turbulent fluctuations from fine-scale incoherent isotropic structures.
- \( \hat{q} \) = Turbulent fluctuations from large-scale coherent anisotropic structures.
- \( q' \) = Acoustic radiation from large-scale anisotropic structures.
- \( q'' \) = Acoustic radiation from fine-scale isotropic structures.

- We substitute this decomposition in the Navier-Stokes equations.
- The resulting equations are rearranged such that the source terms are brought to the right-hand side, and the acoustic fluctuations are kept on the left-hand side.
Spectral Density of Pressure

After some simplifications, we obtain the spectral density of the field variable as

$$S_{k}^\perp(x, \omega) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{m=0}^{4} \sum_{n=0}^{4} q_{g,k}^\perp,m(x; y, \omega)q_{g,k}^\perp,n(x; y + \eta, \omega)R_{m,n}^\perp(y, \eta, \tau)e^{i\omega \tau}d\tau d\eta dy,$$

where

$$R_{m,n}^\perp(y, \eta, \tau) = \langle \Theta_{m}(y, \tau)\Theta_{n}(y + \eta, \tau + \Delta\tau) \rangle = \int_{-\infty}^{\infty} \Theta_{m}(y, \tau)\Theta_{n}(y + \eta, \tau + \Delta\tau)d\Delta\tau.$$

Substituting $k = 4$ to find the spectral density of pressure, we obtain

$$S_{4}^\perp(x, \omega) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{m=0}^{4} \sum_{n=0}^{4} p_{g,m}^\perp,m(x; y, \omega)p_{g,n}^\perp,n(x; y + \eta, \omega)R_{m,n}^\perp(y, \eta, \tau)e^{i\omega \tau}d\tau d\eta dy.$$
Vector Green’s Function Equations

The vector Green’s function

\[ q_{k,g}^\perp = \left[ \rho_{g}^\perp, \mathbf{u}_{i,g}^\perp, p_{g}^\perp \right]^T \]

satisfies

\[
\frac{\partial \rho_{g}^\perp}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho_{u_j,g}^\perp + \rho_{g}^\perp \mathbf{u}_j \right) = \delta(x-y) \delta(t-\tau) \delta_0,
\]

\[
\frac{\partial}{\partial t} \left( \rho_{u_i,g}^\perp + \rho_{g}^\perp \mathbf{u}_i \right) + \frac{\partial}{\partial x_j} \left( \rho_{u_i} \mathbf{u}_{j,g}^\perp + \rho_{u_i,g}^\perp \mathbf{u}_j + \rho_{g}^\perp \mathbf{u}_i \mathbf{u}_j \right) + \frac{\partial p_{g}^\perp}{\partial x_j} \delta_{ij} - \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_{i,g}^\perp}{\partial x_j} + \frac{\partial u_{j,g}^\perp}{\partial x_i} \right) \right] \\
+ \frac{2}{3} \frac{\partial}{\partial x_j} \left[ \mu \frac{\partial u_{k,g}^\perp}{\partial x_k} \right] = \delta(x-y) \delta(t-\tau) \delta_{in},
\]

and

\[
\frac{\partial p_{g}^\perp}{\partial t} + \frac{\gamma - 1}{2} \frac{\partial}{\partial t} \left( \rho_{g}^\perp \mathbf{u}_k \mathbf{u}_k + 2 \rho_{u_k,g}^\perp \mathbf{u}_k \right) + \frac{\gamma - 1}{2} \frac{\partial}{\partial x_j} \left( \rho_{g}^\perp \mathbf{u}_j \mathbf{u}_k + \rho_{u_j,g}^\perp \mathbf{u}_k \mathbf{u}_k + 2 \rho_{u_j} \mathbf{u}_{k,g}^\perp \mathbf{u}_k \right) \\
+ \gamma \frac{\partial}{\partial x_j} \left( \mathbf{u}_{j,g}^\perp p + \mathbf{u}_j p_{g}^\perp \right) - (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \frac{c_p \mu}{Pr} \frac{\partial T_{g}^\perp}{\partial x_j} \right] + (\gamma - 1) \frac{2}{3} \delta_{ij} \frac{\partial}{\partial x_j} \left[ \mu \mathbf{u}_{i,g}^\perp \frac{\partial \mathbf{u}_k}{\partial x_k} + \mu \mathbf{u}_i \frac{\partial \mathbf{u}_{k,g}^\perp}{\partial x_k} \right] \\
- (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \mu \mathbf{u}_{i,g}^\perp \left( \frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i} \right) + \mu \mathbf{u}_i \left( \frac{\partial \mathbf{u}_{i,g}^\perp}{\partial x_j} + \frac{\partial \mathbf{u}_{j,g}^\perp}{\partial x_i} \right) \right] = \delta(x-y) \delta(t-\tau) \delta_{4n}.
\]
Vector Green’s Function for Pressure

The vector Green’s function for pressure is obtained as

\[
\tilde{p}_g^{\perp n} = \left( -\frac{i\omega}{4\pi c_\infty r} + \frac{1}{4\pi r^2} \right) \left( x_i - y_i \right) \exp \left[ \frac{i\omega r}{c_\infty} \right] \delta_{in} - \frac{i\omega}{4\pi c_\infty^2 r} \exp \left[ \frac{i\omega r}{c_\infty} \right] \delta_{4n}.
\]

For the far-field, we can simplify the vector Green’s function to

\[
\tilde{p}_g^{\perp n}(x, y, \omega) = -\frac{i\omega}{4\pi c_\infty r} \exp \left[ \frac{i\omega r}{c_\infty} \right] \left( \frac{x_i}{r} \delta_{in} + \frac{1}{c_\infty} \delta_{4n} \right).
\]

Following Tam and Auriault\textsuperscript{4}, we obtain

\[
\tilde{p}_g^{\perp n}(x, y + \eta, \omega) \approx \tilde{p}_g^{\perp n}(x, y, \omega) \exp \left[ -\frac{i\omega x \cdot \eta}{c_\infty r} \right],
\]

which is a relation for the phase difference between two stream-wise source locations after the far-field assumption has been made.

Aerodynamic Source Terms

- The source terms on the right-hand side are

\[ \Theta_0 = -\frac{\partial \rho}{\partial t} - \frac{\partial \rho u_j}{\partial x_j}, \]

\[ \Theta_i = -\frac{\partial \rho u_i}{\partial t} - \frac{\partial \rho u_i u_j}{\partial x_j} - \frac{\partial p}{\partial x_j} \delta_{ij} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{2}{3} \frac{\partial}{\partial x_j} \left[ \mu \frac{\partial u_k}{\partial x_k} \right], \]

and

\[ \Theta_4 = -\frac{\partial p}{\partial t} - \frac{\gamma - 1}{2} \frac{\partial \rho u_k u_k}{\partial t} - \frac{\gamma - 1}{2} \frac{\partial \rho u_j p}{\partial x_j} - \frac{\gamma - 1}{2} \frac{\partial \rho u_j u_k u_k}{\partial x_j} + (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \frac{c_p \mu}{Pr} \frac{\partial T}{\partial x_j} \right] \]

\[ + (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \mu u_i \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{2}{3} \delta_{ij} \frac{\partial}{\partial x_j} \left[ \mu u_i \frac{\partial u_k}{\partial x_k} \right]. \]

- The sources are the exact Navier-Stokes equations operating on \( q = \bar{q} + \hat{q} + \check{q} \).
Source Terms

A matrix form of all the two-point cross-correlations is given as

$$R^\perp_{m,n} = \begin{bmatrix}
R^\perp_{00} & R^\perp_{01} & R^\perp_{02} & R^\perp_{03} & R^\perp_{04} \\
R^\perp_{10} & R^\perp_{11} & R^\perp_{12} & R^\perp_{13} & R^\perp_{14} \\
R^\perp_{20} & R^\perp_{21} & R^\perp_{22} & R^\perp_{23} & R^\perp_{24} \\
R^\perp_{30} & R^\perp_{31} & R^\perp_{32} & R^\perp_{33} & R^\perp_{34} \\
R^\perp_{40} & R^\perp_{41} & R^\perp_{42} & R^\perp_{43} & R^\perp_{44}
\end{bmatrix}.$$ 

For eg., the two-point cross-correlation for the continuity-continuity equation is written as

$$R^\perp_{0,0} = \langle \Theta_0(y, \tau), \Theta_0(y + \eta, \tau + \Delta \tau) \rangle = \left\langle \left( -\frac{\partial \rho^{(1)}}{\partial \tau} - \frac{\partial \rho^{(1)} u_j^{(1)}}{\partial y_j} \right), \left( -\frac{\partial \rho^{(2)}}{\partial \tau} - \frac{\partial \rho^{(2)} u_m^{(1)}}{\partial y_m} \right) \right\rangle.$$ 

Expanding the above equation, we can write the two-point cross-correlation as

$$R^\perp_{0,0}(y, \eta, \tau) = \left\langle \frac{\partial \rho^{(1)}}{\partial \tau}, \frac{\partial \rho^{(2)}}{\partial \tau} \right\rangle + \left\langle \frac{\partial \rho^{(1)}}{\partial \tau}, \frac{\partial \rho^{(2)} u_m^{(1)}}{\partial y_m} \right\rangle + \left\langle \frac{\partial \rho^{(1)} u_j^{(1)}}{\partial y_j}, \frac{\partial \rho^{(2)}}{\partial \tau} \right\rangle + \left\langle \frac{\partial \rho^{(1)} u_j^{(1)}}{\partial y_j}, \frac{\partial \rho^{(2)} u_m^{(1)}}{\partial y_m} \right\rangle.$$
Following Ribner\textsuperscript{5}, we model these two-point cross-correlation terms as

\[ R_{m,n}^\perp = \mathcal{A} R. \]

Using the model, we expand the first term of the continuity-continuity correlation as

\[
\left\langle \frac{\partial \rho^{(1)}}{\partial \tau} , \frac{\partial \rho^{(2)}}{\partial \tau} \right\rangle = \frac{\partial \rho^{(1)}}{\partial \tau} \frac{\partial \rho^{(2)}}{\partial \tau} + \frac{\partial \rho^{(1)}}{\partial \tau} \frac{\partial \rho^{(2)}}{\partial \tau} \frac{\hat{\rho}^{(2)}}{\hat{\tau}_s^{(2)}} + \frac{\partial \rho^{(1)}}{\partial \tau} \frac{\rho^{(2)}}{\rho_s^{(2)}} + \frac{\rho^{(1)}}{\rho_s^{(1)}} \frac{\partial \rho^{(2)}}{\partial \tau} + \frac{\rho^{(1)}}{\rho_s^{(1)}} \frac{\rho^{(2)}}{\rho_s^{(2)}} \hat{\mathcal{R}}
\]

\[
+ \frac{\hat{\rho}^{(1)}}{\hat{\tau}_s^{(2)}} \hat{\rho}^{(2)} \hat{\mathcal{R}} + \frac{\hat{\rho}^{(1)}}{\hat{\tau}_s^{(2)}} \frac{\partial \rho^{(2)}}{\partial \tau} + \frac{\rho^{(1)}}{\rho_s^{(1)}} \frac{\rho^{(2)}}{\rho_s^{(2)}} \hat{\mathcal{R}} + \frac{\rho^{(1)}}{\rho_s^{(1)}} \frac{\rho^{(2)}}{\rho_s^{(2)}} \hat{\mathcal{R}}.
\]

One of the terms in the energy-energy correlation consists of 6561 decomposed terms!

Elimination of Source Terms based on Physical Mechanism

• After eliminating the viscous terms, the remaining terms are

\[ \Theta_i = -\frac{\partial \rho u_i}{\partial t} - \frac{\partial \rho u_i u_j}{\partial x_j} - \frac{\partial p}{\partial x_j} \delta_{ij}, \]

and

\[ \Theta_4 = -\frac{\partial p}{\partial t} - \frac{\gamma - 1}{2} \frac{\partial u_k u_k}{\partial t} - \frac{\partial u_j p}{\partial x_j} - \frac{\gamma - 1}{2} \frac{\partial u_j u_k u_k}{\partial x_j}. \]

• We also eliminate several terms based on the mathematical properties of two-point cross-correlation.
BBSAN Source Terms

Remaining terms are

\[ R^{(1)} = \left\langle \frac{\partial \rho^{(1)} u_i^{(1)} u_j^{(1)}}{\partial y_j}, \frac{\partial \rho^{(2)} u_i^{(2)} u_m^{(2)}}{\partial y_m} \right\rangle, \]

\[ R^{(2)} = \gamma \left\langle \frac{\partial \rho^{(1)} u_i^{(1)} u_j^{(1)}}{\partial y_j}, \frac{\partial u_m^{(2)} p^{(2)}}{\partial y_m} \right\rangle, \]

\[ R^{(3)} = \frac{\gamma - 1}{2} \left\langle \frac{\partial \rho^{(1)} u_i^{(1)} u_j^{(1)}}{\partial y_j}, \frac{\partial \rho^{(2)} u_m^{(2)} u_n^{(2)} u_n^{(2)}}{\partial y_m} \right\rangle, \]

\[ R^{(4)} = \gamma^2 \left\langle \frac{\partial u_j^{(1)} p^{(1)}}{\partial y_j}, \frac{\partial u_m^{(2)} p^{(2)}}{\partial y_m} \right\rangle, \]

\[ R^{(5)} = \frac{\gamma (\gamma - 1)}{2} \left\langle \frac{\partial u_j^{(1)} p^{(1)}}{\partial y_j}, \frac{\partial \rho^{(2)} u_m^{(2)} u_n^{(2)}}{\partial y_m} \right\rangle, \]

\[ R^{(6)} = \frac{(\gamma - 1)^2}{4} \left\langle \frac{\partial \rho^{(1)} u_j^{(1)} u_k^{(1)} u_k^{(1)}}{\partial y_j}, \frac{\partial \rho^{(2)} u_m^{(2)} u_n^{(2)} u_n^{(2)}}{\partial y_m} \right\rangle. \]

The normalized two-point cross correlation is

\[ \hat{R} = \exp \left[ -\frac{|\tau|}{\tau_s} \right] \exp \left[ -\frac{(\xi - u_c \tau)^2}{l^2} \right] \exp \left[ -\frac{(n^2 + \zeta^2)}{l_\perp^2} \right]. \]
Scaling of BBSAN Source Term\textsuperscript{6}

Over-expanded conditions

\[ \gamma^2 \frac{\partial \tilde{u}_j^{(1)}}{\partial y_j} \frac{\partial \tilde{u}_m^{(2)}}{\partial y_m} \]

\[ \beta = (|M_j^2 - M_m^2|)^{1/2} \]

Under-expanded conditions

\[ \gamma^2 \frac{\partial \tilde{u}_j^{(1)}}{\partial y_j} \frac{\partial \tilde{u}_m^{(2)}}{\partial y_m} \]

\[ \beta = (|M_j^2 - M_m^2|)^{1/2} \]

Spectral Density of Pressure

After integrating and simplifying, we obtain a closed-form prediction model as

$$S_{4}^{\perp}(\mathbf{x}, \omega) = \frac{\gamma^2 \omega^2}{16\pi \sqrt{\pi} c_{\infty}^4 r^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l_{\perp}^2 l_{s} \exp \left[ -\frac{\omega^2 l_{\perp}^2 \sin^2 \theta}{4c_{\infty}^2} \right] \frac{\partial \bar{p}}{\partial y_j}(\mathbf{y}) \hat{u}_j(\mathbf{y})$$

$$\left[ \int_{-\infty}^{\infty} \exp \left[ -l^2 \left( \kappa - \frac{\omega \cos \theta}{c_{\infty}} \right)^2 / 4 \right] \left[ 1 + \left( 1 - M_c \cos \theta + \frac{u_c \kappa}{\omega} \right)^2 \omega^2 \tau_s^2 \right] \frac{\partial \bar{p}}{\partial y_m}(\kappa, y_2, y_3) d\kappa \right] \hat{u}_m(\mathbf{y}) d\mathbf{y},$$

We use the RANS CFD results to estimate the mean quantities, length scales, and time scales.

$$\tau_s = c_T K / \epsilon; \quad l = c_l K^{3/2} / \epsilon; \quad l_{\perp} = c_{\perp} l.$$ 

However, very good prediction results are obtained using LES data with the identified source term and free-space Green’s function\(^7\).

Prediction Results for $M_j = 1.734$

SPL per unit St.

$SPL_{\text{max}} = 130.13$

$SPL_{\text{max}} = 127.94$

$SPL_{\text{max}} = 126.11$

$SPL_{\text{max}} = 127.3$

$SPL_{\text{max}} = 127.31$

$SPL_{\text{max}} = 132.31$

$SPL_{\text{max}} = 130.31$

$SPL_{\text{max}} = 127.3$
Sensitivity Analyses of BBSAN on Model Parameters

- Different parameters in the model such as \( l \), \( l_\perp \), \( \tau_s \), \( \partial p/\partial y_j \), \( \hat{u}_j \), and \( u_c \) are perturbed by 1% and the effect on the change in SPL is plotted.
- The experimental SPL per unit St. is superimposed on the plot.
- The sensitivity of \( \partial p/\partial y_j \) and \( \hat{u}_j \) is constant (0.1 dB) for all Strouhal numbers.
- The sensitivity of \( l \), \( l_\perp \), \( \tau_s \), and \( u_c \) is high at the trough between two BBSAN peaks and decreases at the BBSAN peak.
Sensitivity Analyses of BBSAN on Input Parameters

- We perform the sensitivity analysis of the BBSAN source term using an unheated SMC016 nozzle and a heated biconic nozzle.

- The NPR, TTR and Area Ratio are perturbed by 1% to find the sensitivity of the nozzle at those conditions.

- The SMC016 nozzle is perturbed using a MOC tool developed by Rice$^8$.

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### Table: Perturbed Values for SMC016 and Biconic Nozzles

<table>
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<th>NPR</th>
<th>TTR</th>
<th>NPR</th>
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SMC016 and Biconic Nozzle Comparison

- The biconic nozzle contains shocks even at design conditions.
- Hence, shock-associated noise is present even at design condition.
Unheated SMC016 Nozzle - NPR Variation

- Noise is minimum at design condition, and noise increases if varied from that condition.
- The maximum sensitivity of BBSAN with NPR is 70.41% and correspondingly, the SPL changes by 3.27 dB.
- The axial location shifts by 0.1D when the NPR is perturbed by 1%.

![Graphs showing source term intensity, SPL peak per unit St, and axial source location for SMC016 Nozzle with NPR variation.](image-url)
BBSAN is least sensitive to variation in temperature ratio (1.48%).

A maximum change of 0.13 dB is observed in the SPL when TTR is perturbed at unheated jet condition.

No change is axial source location is observed.
Unheated SMC016 Nozzle - Area Ratio Variation

- BBSAN is most sensitive to variation in area ratio (237.17%).
- The SPL decreases by a maximum of 4 dB for over-expanded conditions and increases by 6.2 dB for under-expanded conditions when the area ratio is decreased.
- If the area ratio is increased, the SPL increases by a maximum of 5 dB for over-expanded conditions and reduces by 2 dB for under-expanded conditions.
- A shift of 2.5D is observed in the axial source locations near the design conditions.
Heated Biconic Nozzle - NPR Variation

- BBSAN is present at design condition also for biconic nozzles.
- The biconic nozzle is not as sensitive as the SMC016 nozzle when the NPR, TTR and area ratio are perturbed.
- Maximum sensitivity for 1% perturbation in NPR is 20.22%.
- The maximum change in SPL is less than 0.5 dB for most cases.
- The source location shifts by 0.1D for the under-expanded conditions.
Heated Biconic Nozzle - TTR Variation

- The source term intensity varies by a maximum of 0.36% and the SPL varies by a maximum of 0.2 dB when the TTR is perturbed.
- No variation in the source locations is observed when the TTR is perturbed.
Heated Biconic Nozzle - Area Ratio Variation

- Maximum sensitivity for 1% perturbation in area ratios is 21.32%.
- The SPL varies by a maximum of 1.5 dB when the area ratio is perturbed.
- As the shock-cell structure is always present in the jet plume, the reduction of noise does not happen as rapidly as the MOC nozzle.
- Very little change is observed in the axial source location (0.05D) for biconic nozzle.
Effect of Boundary Layer

- We compare the boundary layer profile at the internal exit of the nozzle using backward step, forward step, and extended region in the converging section.
- Maximum reduction of source intensity (3.29%) is observed for backward step for SMC016 nozzle at the NPR=5.2 condition.
- The maximum reduction for SMC016 nozzle is 41.6% and for the biconic nozzle is 2.07%.
Preliminary Effect of Facets

- The biconic nozzle, modified with 12 facets, is used.
- We use different length scales in the cross-stream direction to modify the BBSAN model. We obtain

\[
S_4^+(x, \omega) = \frac{\gamma^2 \omega^2}{16\pi^2 \sqrt{\pi} c_\infty^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l_1 l_2 l_3 \tau_s \exp \left[ -\frac{\omega^2}{4\alpha^2_\infty} \sin^2 \theta \left( l_2^2 \cos^2 \phi + l_3^2 \sin^2 \phi \right) \right] \frac{\partial p}{\partial y_j}(y) \hat{u}_j(y)
\]

\[
\int_{-\infty}^{\infty} \exp \left[ -\frac{l_1^2}{4} \left( \kappa - \frac{\omega \cos \theta}{c_\infty} \right)^2 / 4 \right] \frac{\partial p}{\partial y_m}(\kappa, y_2, y_3) \, d\kappa \frac{\partial p}{\partial y_m}(\kappa, y_2, y_3) \, d\kappa
\]

- The source term intensity is increased by \( 147.64\% \) at \( M_j = 1.828 \).
- We are currently evaluating the change in SPL using the developed model from the baseline biconic nozzle under heated conditions.
Summary and Conclusions

- **Source Term**
  - The source term responsible for BBSAN i.e. $\gamma^2 \left< \hat{u}_j^{(1)} \frac{\partial p^{(1)}}{\partial y_j}, \hat{u}_m^{(2)} \frac{\partial p^{(2)}}{\partial y_m} \right>$ has been determined from Navier-Stokes equations.
  - A closed-form prediction model has been developed using the identified source term.

- **Sensitivity Analyses of Model Parameters**
  - Maximum sensitivity at the BBSAN peak is less than 0.1 dB for all the model parameters.

- **Sensitivity analyses of input conditions such as NPR, TTR, and Area Ratio**
  - The effect on the source term intensity, peak SPL and axial source location is performed.
  - Biconic nozzle is less sensitive compared to SMC016 nozzle.
  - Area Ratio is the most sensitive and TTR is the least sensitive.

- **Effect of Boundary Layer and Facets**
  - The biconic nozzle is less sensitive ($< 0.1$ dB) with the perturbation in the boundary layer.
  - For SMC016 nozzle, the BBSAN increases for over-expanded conditions (2.5 dB) and decreases for under-expanded conditions (2 dB).
  - The source term intensity increased by a maximum of 147.64% for faceted nozzle when compared with the biconic nozzle.
Thank You for your Attention!
Questions?
Backup Slides
Broadband Shock-Associated Noise

- A shock-cell structure is generated in the jet exhaust when the jet is operating at off-design conditions.
- Harper-Bourne and Fisher\(^9\) observed that BBSAN scales as \(\beta^4\), where \(\beta = \left( |M_j^2 - M_d^2| \right)^{1/2} \).

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10 Tam and Tanna, “Shock Associated Noise of Supersonic Jets from Convergent-Divergent Nozzles”.

OASPL for a supersonic jet\(^10\)
Governing Equations and Assumptions

The conservation of mass, momentum and energy equations are written as

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0,
\]

\[
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j},
\]

and

\[
\frac{\partial \rho e_0}{\partial t} + \frac{\partial \rho u_j e_0}{\partial x_j} = -\frac{\partial q_j}{\partial x_j} + \frac{\partial u_j \sigma_{ij}}{\partial x_j}.
\]

- **Constitutive relationship**

  \[
  \sigma_{ij} = \tau_{ij} - p \delta_{ij}
  \]

  \[
  \tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij}
  \]

- **Fourier’s law**

  \[
  q_j = -c_p \mu \frac{\partial T}{\partial x_j}
  \]

- **Calorically perfect gas equations**

  \[
  p = \rho \mathcal{R} T
  \]

  \[
  e_0 = e + \frac{1}{2} u_k u_k; \quad e = c_v T
  \]

  \[
  \mathcal{R} = c_p - c_v; \quad \gamma = \frac{c_p}{c_v}
  \]
The Navier-Stokes Equations

The Navier-Stokes equations are written as

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \]

\[ \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_j} \delta_{ij} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{2}{3} \frac{\partial}{\partial x_j} \left[ \mu \frac{\partial u_k}{\partial x_k} \right], \]

and

\[ \frac{\partial p}{\partial t} + \frac{\gamma - 1}{2} \frac{\partial \rho u_k u_k}{\partial t} + \frac{\gamma - 1}{2} \frac{\partial \rho u_j u_k u_k}{\partial x_j} = -\gamma \frac{\partial u_j p}{\partial x_j} + (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \frac{c_p \mu}{Pr} \frac{\partial T}{\partial x_j} \right] + (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \mu u_i \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - (\gamma - 1) \frac{2}{3} \delta_{ij} \frac{\partial}{\partial x_j} \left[ \mu u_i \frac{\partial u_k}{\partial x_k} \right]. \]
Correlations and Fourier Transforms

The auto-correlation function is defined as

\[
< q_k^\perp(x, t) q_k^\perp(x, t + \tau^\dagger) > = \int_{-\infty}^{\infty} q_k^\perp(x, t) q_k^\perp(x, t + \tau^\dagger) dt
\]

\[
= \int_{-\infty}^{\infty} \tilde{q}_k^\perp(x, \omega) \tilde{q}_k^\perp(x, \omega) \exp[-i\omega \tau^\dagger] d\omega.
\]

while the forward and inverse Fourier transforms are defined as

\[
\tilde{q}_k^\perp(x, \omega) = \int_{-\infty}^{\infty} q_k^\perp(x, t) \exp[-i\omega t] dt
\]

and

\[
q_k^\perp(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{q}_k^\perp(x, \omega) \exp[i\omega t] d\omega.
\]
Radiating Acoustic Terms

The propagators \(( q^\perp = q' + q'' )\) on the left-hand side are linearized and are written as

\[
\frac{\partial \rho^\perp}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho u_j^\perp + \rho^\perp u_j \right) = \Theta_0,
\]

\[
\frac{\partial}{\partial t} \left( \rho u_i^\perp + \rho^\perp u_i \right) + \frac{\partial}{\partial x_j} \left( \rho u_i u_j^\perp + \rho u_i^\perp u_j + \rho^\perp u_i u_j \right) + \frac{\partial p^\perp}{\partial x_j} \delta_{ij} \\
- \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i^\perp}{\partial x_j} + \frac{\partial u_j^\perp}{\partial x_i} \right) \right] + \frac{2}{3} \frac{\partial}{\partial x_j} \left[ \mu \frac{\partial u_k^\perp}{\partial x_k} \right] = \Theta_i,
\]

and

\[
\frac{\partial p^\perp}{\partial t} + \frac{\gamma - 1}{2} \frac{\partial}{\partial t} \left( \rho^\perp u_k u_k + 2 \rho u_k^\perp u_k \right) + \frac{\gamma - 1}{2} \frac{\partial}{\partial x_j} \left( \rho^\perp u_j u_k u_k + \rho u_j^\perp u_k u_k + 2 \rho u_j u_k^\perp u_k \right) \\
+ \gamma \frac{\partial}{\partial x_j} \left( u_j^\perp p + u_j p^\perp \right) - (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \frac{c_p \mu}{Pr} \frac{\partial T^\perp}{\partial x_j} \right] + (\gamma - 1) \frac{2}{3} \delta_{ij} \frac{\partial}{\partial x_j} \left[ \mu u_i^\perp \frac{\partial u_k}{\partial x_k} + \mu u_i \frac{\partial u_k^\perp}{\partial x_k} \right] \\
- (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \mu u_i^\perp \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \mu u_i \left( \frac{\partial u_i^\perp}{\partial x_j} + \frac{\partial u_j^\perp}{\partial x_i} \right) \right] = \Theta_4.
\]
Spectral Density

Using a convolution integral, we obtain

$$q_{k}^{\perp}(x, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=0}^{4} q_{g,k}^{\perp,j}(x, t; y, \tau) \Theta_{j}(y, \tau) d\tau dy.$$ 

The spectral density is obtained from the auto-correlation of field-variable as

$$S_{k}^{\perp}(x, \omega) = \int_{-\infty}^{\infty} < q_{k}^{\perp}(x, t) q_{k}^{\perp}(x, t + \tau^{\dagger}) > \exp[i\omega\tau^{\dagger}] d\tau^{\dagger}.$$
Simplifying the Spectral Density of Acoustic Variables

Using the definition of forward Fourier transform, we obtain

\[ \tilde{q}_k^\perp(x, \omega) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=0}^{4} q_{g,k}^\perp(x, t; y, \tau) \Theta_j(y, \tau) \exp[-i\omega t] dt d\tau dy. \]

Substituting the above result in the formula of spectral density, we obtain

\[ S_k^\perp(x, \omega) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{m=0}^{4} q_{g,k}^{*,\perp,m}(x, t; y, \tau) \Theta_m(y, \tau) \exp[i\omega t] dt d\tau dy \right] \]

\[ \left[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{n=0}^{4} q_{g,k}^{\perp,n}(x, t'; z, \tau') \Theta_n(z, \tau') \exp[-i\omega t'] dt' d\tau' dz \right] \exp[-i\omega \tau^\dagger] d\omega \exp[i\omega \tau^\dagger] d\tau^\dagger. \]
Simplifying using the definitions of forward and inverse Fourier transforms, we obtain

\[
S_{\perp}^k(x, \omega) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[ \sum_{m=0}^{4} \sum_{n=0}^{4} \tilde{q}_{g,k}^*, m(x; y, \omega) \tilde{q}_{g,k}^\perp, n(x; z, \omega) \int_{-\infty}^{\infty} \Theta_m(y, \tau) \Theta_n(z, \tau') d\tau' \right] \\
\times \int_{-\infty}^{\infty} \Theta_m(y, \tau) \Theta_n(y + \eta, \tau + \Delta \tau) d\Delta \tau \exp[i\omega \tau] d\tau d\eta dy.
\]

Using the relations, \( \eta = z - y \) and \( \tau' - \tau = \Delta \tau \), we simplify the above equation to obtain

\[
S_{\perp}^k(x, \omega) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[ \sum_{m=0}^{4} \sum_{n=0}^{4} \tilde{q}_{g,k}^*, m(x; y, \omega) \tilde{q}_{g,k}^\perp, n(x; y + \eta, \omega) \right] \\
\times \int_{-\infty}^{\infty} \Theta_m(y, \tau) \Theta_n(y + \eta, \tau + \Delta \tau) d\Delta \tau \exp[i\omega \tau] d\tau d\eta dy.
\]
Vector Green’s Function for Pressure

- We assume that the environment is quiescent and the refraction effects in the sideline direction of the jet are negligible.
- We perform mathematical operations on the linearized momentum and energy Green’s function equations and subtract them to obtain a wave equation in terms of pressure.
- The components of the vector Green’s function of pressure are obtained using the free space Green’s function of Helmholtz equation.
- Finally, we make a far-field assumption to simplify the vector Green’s function.
Vector Green’s Function for Pressure

We subtract the time derivative of energy Green’s function equation from the momentum Green’s function equation and obtain

\[
\frac{\partial^2 p_{\perp n}}{\partial t^2} - c^2 \frac{\partial^2 p_{\perp n}}{\partial x_i^2} = \frac{\partial}{\partial t} [\delta(x - y)\delta(t - \tau)\delta_{4n}] - c^2 \frac{\partial}{\partial x_i} [\delta(x - y)\delta(t - \tau)\delta_{4n}].
\]

We perform the Fourier transform with respect to time on the previous equation and obtain

\[
\omega^2 \tilde{p}_{\perp n} + \frac{\partial^2 \tilde{p}_{\perp n}}{\partial x_i^2} = i\omega \delta(x - y)\delta_{4n} + \frac{\partial}{\partial x_i} \delta(x - y)\delta_{4n}.
\]

The free-space Green’s function satisfying the Helmholtz equation

\[
\frac{\omega^2}{c^2} g(x, z, \omega) + \frac{\partial^2 g(x, z, \omega)}{\partial x_i^2} = \delta(x - z)
\]

is given by

\[
g(x, z, \omega) = -\frac{\exp[i\omega|x - z|/c_\infty]}{4\pi|x - z|}.
\]
The vector Green’s function for pressure from is derived as

$$\tilde{p}_g^{\perp n}(x, y, \omega) = \int\int\int_{-\infty}^{\infty} g(x, z, \omega) \left[ \frac{i\omega}{c^2} \delta(z - y)\delta_{4n} + \frac{\partial}{\partial z_i} \delta(z - y)\delta_{in} \right] dz$$

$$= \frac{i\omega}{c^2} \int\int\int_{-\infty}^{\infty} g(x, z, \omega) \delta(z - y)\delta_{4n} dz + \int\int\int_{-\infty}^{\infty} g(x, z, \omega) \frac{\partial}{\partial z_i} \left[ \delta(z - y)\delta_{in} \right] dz$$

$$= \frac{i\omega}{c^2} g(x, y, \omega)\delta_{4n} - \int\int\int_{-\infty}^{\infty} \frac{\partial}{\partial z_i} [g(x, z, \omega)] \delta(z - y)\delta_{in} dz$$

$$= \frac{i\omega}{c^2} g(x, y, \omega)\delta_{4n} - \frac{\partial}{\partial y_i} [g(x, y, \omega)] \delta_{in}.$$
Deriving the Spectral Density of Pressure

Substituting the vector Green’s function and the source terms, we obtain

\[
S_4^\perp(x, \omega) = \frac{\gamma^2 \omega^2}{16 \pi^2 c_\infty^4 \tau^2} \int \int \int \frac{\partial \tilde{p}^{(1)}}{\partial y_j} \frac{\partial \tilde{p}^{(2)}}{\partial y_m} \hat{u}_j^{(1)} \hat{u}_m^{(2)} \hat{R} \exp \left[ - \frac{i \omega x \cdot \eta}{c_\infty r} \right] \exp [i \omega \tau] d\tau d\eta dy.
\]

Following Morris and Miller\(^\text{11}\), and taking Fourier transform of one gradient of pressure term,

\[
S_4^\perp(x, \omega) = \frac{\gamma^2 \omega^2}{32 \pi^3 c_\infty^4 r^2} \int \int \int \frac{\partial \tilde{p}(y)}{\partial y_j} \frac{\partial \tilde{p}(\kappa, y_2, y_3)}{\partial y_m} \hat{u}_j(y) \hat{u}_m(y) \exp [i \kappa \xi] \times \hat{R} \exp \left[ i \omega \left( \tau - \frac{x \cdot \eta}{c_\infty r} \right) \right] d\kappa d\tau d\eta dy,
\]

We integrate the above equation with respect to \(\tau\) and \(\eta\)

\[
\int \int \hat{R} \exp [i \kappa \xi] \exp \left[ i \omega \left( \tau - \frac{x \cdot \eta}{c_\infty r} \right) \right] d\tau d\eta = \int \int \int \exp \left[ - \frac{\eta^2}{l_\perp^2} - \frac{i \omega \eta}{c_\infty} \sin \theta \cos \phi \right] d\eta \int \int \exp \left[ - \frac{\zeta^2}{l_\perp^2} - \frac{i \omega \zeta}{c_\infty} \sin \theta \sin \phi \right] d\zeta
\]

\[
\times \int \int \exp \left[ - \frac{\xi^2}{l_\perp^2} + \frac{2 u_c \tau \xi}{l^2} - \frac{i \omega \xi}{c_\infty} \cos \theta + i \kappa \xi \right] d\xi \exp \left[ \frac{i \omega \tau - \frac{\tau}{\tau_s} - \frac{u_c^2 \tau^2}{l^2} \right] d\tau.
\]

\(^\text{11}\) Morris and Miller, “Prediction of Broadband Shock-Associated Noise Using Reynolds-Averaged Navier-Stokes Computational Fluid Dynamics".
Spectral Density of Pressure

We previously obtained the spectral density of pressure as

\[ S_4^\perp(x, \omega) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{m=0}^{4} \sum_{n=0}^{4} p_g^{*, \perp, m}(x; y, \omega) p_g^{\perp, n}(x; y + \eta, \omega) R_{m,n}^\perp(y, \eta, \tau) \exp[i\omega \tau] \, d\tau \, d\eta \, dy. \]

where \( R_{m,n}^\perp \) is the two-point cross-correlation between source terms, and is written as

\[ R_{m,n}^\perp(y, \eta, \tau) = \langle \Theta_m(y, \tau) \Theta_n(y + \eta, \tau + \Delta \tau) \rangle = \int_{-\infty}^{\infty} \Theta_m(y, \tau) \Theta_n(y + \eta, \tau + \Delta \tau) \, d\Delta \tau. \]

The two point cross – correlation term consists of correlation between continuity – continuity, continuity – momentum, continuity – energy, momentum – momentum, momentum – energy, and energy – energy equations.
Computational Fluid Dynamics is used to evaluate the source terms.

Reynolds-averaged Navier-Stokes (RANS) $K - \Omega$ turbulence model is used for the current work.

Fully Unstructured Navier-Stokes (FUN3D)$^{12}$ developed by NASA Langley Research Center is used.

We validate our results with the experimental measurements of Panda and Seasholtz$^{13}$ using two different cases with $M_d = 1.4$ and $M_d = 1.8$.

---

Computational Domain and Mesh

- Domain: 50D in the cross-stream direction and 100D in the stream-wise direction.
- Around 4-5 million cells in quarter domain.
- $y^+ \approx 1.5$.
- Takes 700-800 wall clock hours for one simulation.
Validation of the Jet Centerline Profiles

\[ M_j = 1.395 \]

\[ M_j = 1.795 \]

\[ \text{Experimental - Panda} \quad \text{RANS Simulation - FUN3D} \]
Validation of the Jet Radial Profiles

\[ M_j = 1.395 \]

\[ \frac{u}{u_j} + n \]

\[ \frac{r}{D} \]

\[ \frac{x}{D} = 2 \]

\[ 4 \]

\[ 6 \]

\[ 8 \]

\[ 10 \]

\[ 12 \]

\[ \frac{T}{T_j} + n \]

\[ \frac{r}{D} \]

\[ \frac{x}{D} = 2 \]

\[ 4 \]

\[ 6 \]

\[ 8 \]

\[ 10 \]

\[ 12 \]

\[ 14 \]

\[ \frac{T}{T_j} + n \]

\[ \frac{r}{D} \]

\[ \frac{x}{D} = 2 \]

\[ 4 \]

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\[ 12 \]

\[ 14 \]

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\[ \frac{r}{D} \]

\[ \frac{x}{D} = 2 \]

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\[ 14 \]

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\[ \frac{x}{D} = 2 \]

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\[ 14 \]

\[ \frac{T}{T_j} + n \]

\[ \frac{r}{D} \]

\[ \frac{x}{D} = 2 \]

\[ 4 \]

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\[ 12 \]

\[ 14 \]

\[ \frac{T}{T_j} + n \]

\[ \frac{r}{D} \]

\[ \frac{x}{D} = 2 \]

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\[ \frac{T}{T_j} + n \]

\[ \frac{r}{D} \]

\[ \frac{x}{D} = 2 \]

\[ 4 \]

\[ 6 \]

\[ 8 \]

\[ 10 \]

\[ 12 \]

\[ 14 \]

\[ \frac{T}{T_j} + n \]

\[ \frac{r}{D} \]
Elimination of Source Terms based on Physical Mechanism

- The momentum and energy source terms are written as

\[ \Theta_i = -\frac{\partial \rho u_i}{\partial t} - \frac{\partial \rho u_i u_j}{\partial x_j} - \frac{\partial p}{\partial x_j} \delta_{ij} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{2}{3} \frac{\partial}{\partial x_j} \left[ \mu \frac{\partial u_k}{\partial x_k} \right], \]

and

\[ \Theta_4 = -\frac{\partial p}{\partial t} - \gamma - 1 \frac{\partial \rho u_k u_k}{\partial t} - \gamma \frac{\partial u_j p}{\partial x_j} - \gamma - 1 \frac{\partial \rho u_j u_k u_k}{\partial x_j} + (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \frac{c_p \mu \partial T}{\rho r \partial x_j} \right] \\
+ (\gamma - 1) \frac{\partial}{\partial x_j} \left[ \mu u_i \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{2}{3} \delta_{ij} \frac{\partial}{\partial x_j} \left[ \mu u_i \frac{\partial u_k}{\partial x_k} \right]. \]

- Various researchers\textsuperscript{14,15,16} found that viscous terms can be neglected for the sound generation of BBSAN.

\textsuperscript{14} Lighthill, "On Sound Generated Aerodynamically I. General Theory".
Scaling of BBSAN Source Term

- Term 1, 3 and 6 are independent of $\beta$.
- Term 2, 5 scale as $\beta^2$.
- The only term that scales as $\beta^4$ is Term-4 i.e. the product of gradient of pressure and the anisotropic velocity.
- From the scaling analysis, it could be inferred that any bi-product of the gradient of mean pressure term can scale as $\beta^4$. Hence, we perform the scaling analysis of the term

$$R^{(7)} = \gamma^2 \frac{\partial p^{(1)}}{\partial y_j} \frac{\partial p^{(2)}}{\partial y_m},$$

Note that although this term is present in the momentum equation, the two-point cross-correlation of this term is zero, because of the presence of only mean quantities. The term-7 roughly scales as $\beta^5$. Hence, the only term that scales as $\beta^4$ is Term-4.
Scaling of BBSAN Source Term

Over-expanded conditions

Under-expanded conditions

\[ \beta = (|M_j^2 - M_d^2|)^{1/2} \]

Normalized Intensity

\begin{align*}
\beta = (|M_j^2 - M_d^2|)^{1/2} \\
\text{Normalized Intensity} &
\end{align*}

Trushant K. Patel and Steven A. E. Miller

Sensitivity of BBSAN based on Navier-Stokes Equations

January 07, 2020
Scaling of BBSAN Source Term

- Term 1, 3 and 6 are independent of $\beta$.
- Term 2, 5 scale as $\beta^2$.
- The only term that scales as $\beta^4$ is Term-4 i.e. the product of gradient of pressure and the anisotropic velocity.

\[ R^{(4)} = \gamma^2 \left\langle \frac{\partial u_j^{(1)} p^{(1)}}{\partial y_j}, \frac{\partial u_m^{(2)} p^{(2)}}{\partial y_m} \right\rangle \rightarrow \gamma^2 \frac{\partial p^{(1)}}{\partial y_j} \frac{\partial p^{(2)}}{\partial y_m} \hat{u}_j^{(1)} \hat{u}_m^{(2)} \hat{R}. \]
Deriving the Spectral Density of Pressure

The spectral density of pressure is given as

$$S_{4}^{\perp}(x, \omega) = \sum_{m=0}^{4} \sum_{n=0}^{4} p_{g}^{\perp,m}(x; y, \omega)p_{g}^{\perp,n}(x; y + \eta, \omega)R_{m,n}^{\perp}(y, \eta, \tau)\exp[i\omega\tau]d\tau d\eta dy.$$ 

The vector Green’s function is

$$\tilde{p}_{g}^{\perp,n}(x, y, \omega) = \frac{-i\omega}{4\pi c_{\infty}r}\exp\left[\frac{i\omega r}{c_{\infty}}\right]\left(\frac{x_i}{r}\delta_{in} + \frac{1}{c_{\infty}}\delta_{4n}\right).$$

The source term for BBSAN is

$$R^{(4)} = \gamma^{2}\frac{\partial p^{(1)}}{\partial y_{j}}\frac{\partial p^{(2)}}{\partial y_{m}}\hat{u}_{j}^{(1)}\hat{u}_{m}^{(2)}\hat{R},$$

where the normalized two-point cross-correlation is modeled following Ribner\textsuperscript{17}

$$\hat{R} = \exp\left[-\frac{|\tau|}{\tau_{s}}\right]\exp\left[-\frac{(\xi - u_{c}\tau)^2}{l^{2}}\right]\exp\left[-\frac{(\eta^{2} + \zeta^{2})}{l_{\perp}^{2}}\right],$$

\textsuperscript{17} Ribner, “Theory of Two-Point Correlations of Jet Noise”.

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Anisotropic Quantities

The composite energy spectrum\(^{18}\) for velocity can be modeled as

\[
E_u(\kappa) = c_u \kappa^{2/3} \kappa^{-5/3} f_{L,u}(\kappa L) f_{\eta,u}(\kappa \eta).
\]

The anisotropic velocities are approximated using the work of Ishihara et al.\(^ {19}\)

\[
\hat{E}_u(\kappa) = \hat{c}_u \kappa^{1/3} \kappa^{-7/3} f_{L,u}(\kappa L) f_{\eta,u}(\kappa \eta),
\]

where

\[
f_{L,u}(\kappa L) = \left[ \frac{\kappa L}{\sqrt{(\kappa L)^2 + 6.78}} \right]^{11/3},
\]

\[
f_{\eta,u}(\kappa \eta) = \exp \left[ -2.1 \left( \left[ (\kappa \eta)^4 + 0.0256 \right]^{1/4} - 0.4 \right) \right].
\]

The velocities are modeled as

\[
\hat{u}_i(y, \tau) = \left[ c_i \int_{\kappa_1}^{\kappa_2} \hat{E}_u(y, \kappa, \tau) d\kappa \right]^{1/2}
\]


Calibration of the model

- We calibrate our model with the experimental measurements of the SHJAR database\(^\text{20}\).
- The observers are located at 100D from the nozzle exit.
- We calibrate our constants using the unheated \(M_j = 1.734\) case for the observer angle at \(\theta = 90^\circ\).
- The calibrated constants are
  \[
  \tau_s = 1.75K/\epsilon
  \]
  \[
  l = 0.75K^{3/2}/\epsilon
  \]
  \[
  l_\perp = 0.33l
  \]
- The same calibrated constants are now used to predict the noise at various operating conditions and different observer angles.

\(^{20}\) Brown and Bridges, “Small Hot Jet Acoustic Rig Validation”.
Results for $M_j = 1.294$

- $SPL_{\text{max}} = 127.58$
- $SPL_{\text{max}} = 131.99$
- $SPL_{\text{max}} = 131.15$
- $SPL_{\text{max}} = 128.23$
- $SPL_{\text{max}} = 123.94$
- $SPL_{\text{max}} = 121.91$
- $SPL_{\text{max}} = 121.91$
- $SPL_{\text{max}} = 128.23$
- $SPL_{\text{max}} = 131.15$
- $SPL_{\text{max}} = 131.99$
- $SPL_{\text{max}} = 127.58$

- $50 \text{ dB}$

- Source location for maximum BBSAN intensity

- BBSAN source strength

- Shock detection quantity

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Results for $M_j = 1.381$

\[10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1\]

- **SPL per unit St.**
- **SPL\text{\text{max}} = 130**
- **SPL\text{\text{max}} = 131.33**
- **SPL\text{\text{max}} = 122.01**
- **SPL\text{\text{max}} = 121.06**
- **SPL\text{\text{max}} = 122.42**
- **SPL\text{\text{max}} = 126.61**
- **SPL\text{\text{max}} = 129.64**
- **SPL\text{\text{max}} = 119.66**
Results for $M_j = 1.611$

SPL per unit $St.$

$\theta = 70^\circ$

$\theta = 80^\circ$

$\theta = 90^\circ$

$\theta = 100^\circ$

$\theta = 110^\circ$

$\theta = 120^\circ$

$\theta = 130^\circ$

$SPL_{\max} = 119.79$

$SPL_{\max} = 121.12$

$SPL_{\max} = 121.21$

$SPL_{\max} = 121.62$

$SPL_{\max} = 122.0$

$SPL_{\max} = 122.48$

$SPL_{\max} = 122.82$

$SPL_{\max} = 123.38$

$SPL_{\max} = 123.62$

$SPL_{\max} = 124.03$

$SPL_{\max} = 124.38$

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$SPL_{\max} = 170.38$

$SPL_{\max} = 170.82$
Results for $M_j = 1.8282$

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**Graphs and Data**

- **Figure (a):**
  - Source location for maximum BBSAN intensity
  - BBSAN source strength
  - Shock detection quantity

- **Figure (b):**
  - Spectral Density
  - Spectral Density

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**Textual Content**

- **Results:**
  - $SPL_{max} = 129.44$
  - $SPL_{max} = 131.54$
  - $SPL_{max} = 130.53$
  - $SPL_{max} = 129.51$
  - $SPL_{max} = 130.25$
  - $SPL_{max} = 129.53$
  - $SPL_{max} = 129.44$

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**Additional Information**

- **Legend:**
  - Prediction
  - Morris-Miller model
  - Experimental

---

**Source Information**

- **Trushant K. Patel and Steven A. E. Miller**
- **Title:** Sensitivity of BBSAN based on Navier-Stokes Equations
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Modification of Model

- The current model will not yield accurate results for non-axisymmetric nozzles and for certain noise reduction techniques because of the current assumptions.
- We will modify the model by assuming different length scales in the cross-stream directions as

$$\hat{R} = \exp \left[ -\frac{|\tau|}{\tau_s} \right] \exp \left[ -\frac{(\xi - u_c\tau)^2}{l_1^2} \right] \exp \left[ -\frac{\eta^2}{l_2^2} - \frac{\zeta^2}{l_3^2} \right].$$

- The length scales can then be determined by using a RANS Reynolds Stress model by using

$$l_1 = c_l \frac{(u'u')^{1/2}}{\Omega},$$
$$l_2 = c_l \frac{(v'v')^{1/2}}{\Omega},$$
$$l_3 = c_l \frac{(w'w')^{1/2}}{\Omega}.$$
BBSAN Reduction

We will analyse the change in source statistics and its correlation with the radiating noise in the far-field, when different noise reduction mechanisms are used.

- Nozzle Corrugations
- Fluidic Injections
- Porous Surfaces
- Beveled nozzles
- Chevrons

---


Large-Scale Noise Source Term

- The remaining source terms are
  \[ \Theta_i = -\frac{\partial \rho u_i}{\partial t} - \frac{\partial \rho u_i u_j}{\partial x_j} - \frac{\partial p}{\partial x_j} \delta_{ij}, \]
  and
  \[ \Theta_4 = -\frac{\partial p}{\partial t} - \frac{\gamma - 1}{2} \frac{\partial u_k u_k}{\partial t} - \gamma \frac{\partial u_j p}{\partial x_j} - \frac{\gamma - 1}{2} \frac{\partial u_j u_k u_k}{\partial x_j}. \]

- We will identify the source term responsible for large-scale noise using the scaling laws of Lighthill\textsuperscript{26} and Ffowcs-Williams\textsuperscript{27} at shallow angles to the jet axis.

- The source terms will also be compared with the scaling of noise from SHJAR database.

26 Lighthill, “On Sound Generated Aerodynamically I. General Theory”