# Toward a Nonlinear Acoustic Analogy: Turbulence as a Source of Sound and Nonlinear Propagation

Steven A. E. Miller
The National Aeronautics and Space Administration

NASA Technical Working Group
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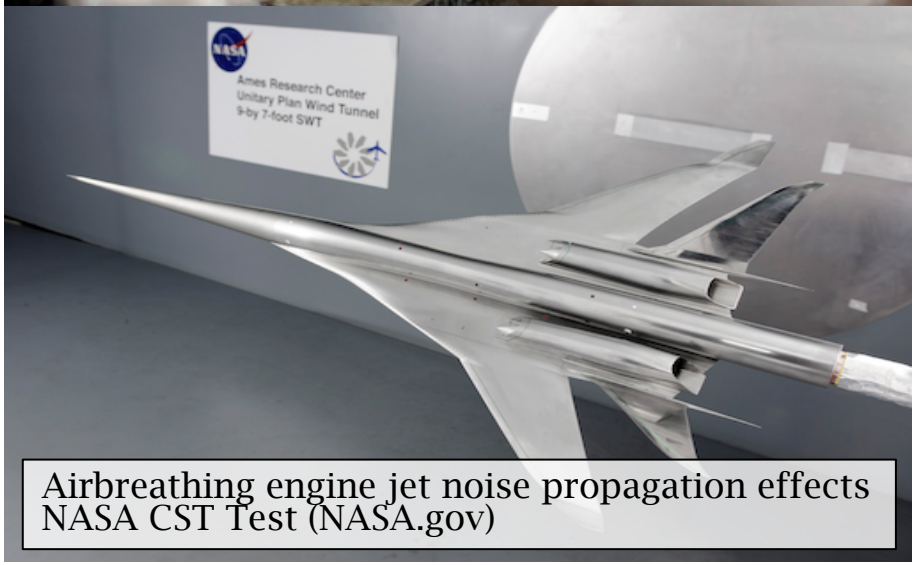
- NASA Advanced Air Vehicles Program Commercial Supersonic Technology Project
- Brian Howerton – NASA Langley – measurements from NASA Normal Incidence Tube
- Emily Mazur – NASA 2012 Intern - evaluated Blackstock bridging function
- Many previous curious researchers

# Introduction

# Turbulence and Nonlinear Propagation

- Aerospace vehicles produce turbulence
- Sound propagates nonlinearly if turbulence is highly intense
- Intense noise is harmful to the vehicle and environment

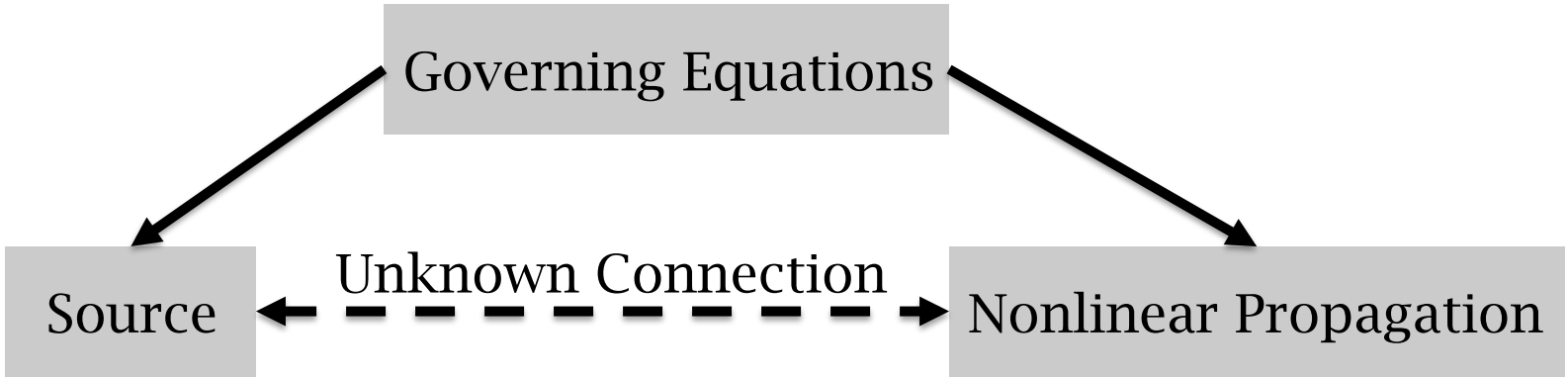
Rocket noise has nonlinear propagation effects
NASA SMAT Test (NASA.gov)




Airbreathing engine jet noise propagation effects
NASA CST Test (NASA.gov)

# Turbulence and Nonlinear Propagation

- Understand different mathematical models of sound generation and propagation
- Relate the governing equations to sound generation and propagation
- Show a mathematical connection between sound generation (acoustic analogy) and sound propagation (Burgers' equation)



# Mathematical Models

# Claude-Louis Navier and George Gabriel Stokes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \qquad \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{\partial \rho e_o}{\partial t} + \frac{\partial \rho u_j e_o}{\partial x_j} = -\frac{\partial u_j p}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \frac{\partial u_i \tau_{ij}}{\partial x_j}$$

Navier

Claude-Louis Marie Henri Navier

- 1735-1836
- French
- Professor at École Nationale des Ponts et Chaussées
- Known for elasticity and structural engineering

Sir George Stokes

- 1819-1903
- Irish
- Lucasian Professor
- Fluids, Optics, Chemistry
- Politics and Theology

Stokes

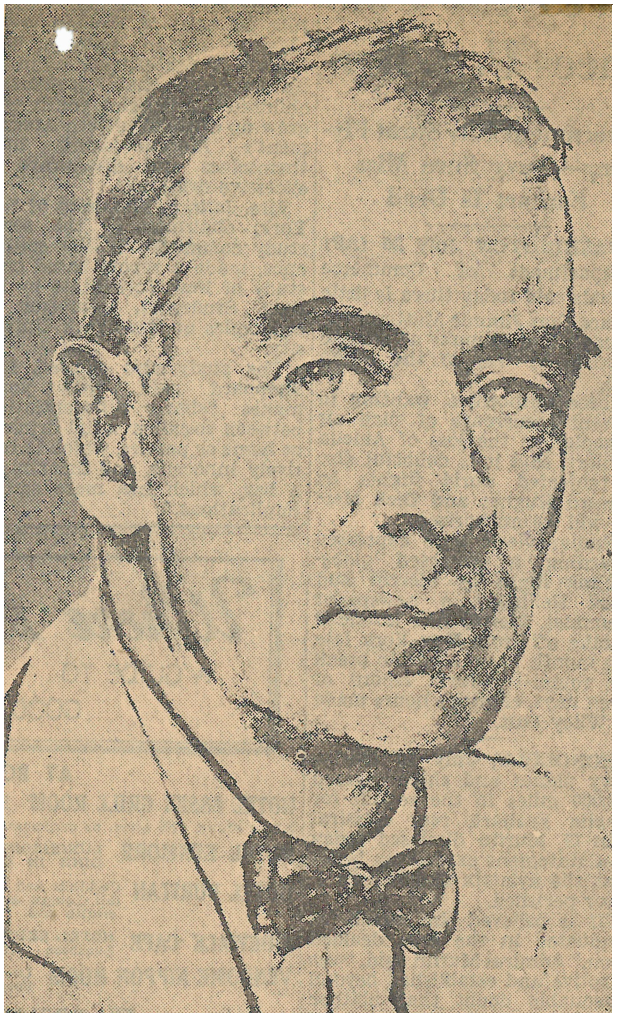
# Richard D. Fay

Governing Equation (Conjecture)

$$c_\infty^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial x}^{\gamma+1} \left[ \frac{\partial^2 y}{\partial t^2} - \frac{4\mu}{3\rho_\infty} \frac{\partial}{\partial t} \left( \frac{\partial^2 y}{\partial x^2} \right) \right]$$

- Governs shocked one-dimensional finite amplitude waves
- *y* is particle displacement
- Solution via assumptions
  - Periodic
  - *dy / dx* is Fourier series
  - Substitute and solve Fourier series

**Fay, R. D.**, "Plane Sound Waves of Finite Amplitude," Journal of the Acoustical Society of America, Vol. 3, No. 9, 1931, pp. 222–241. doi:10.1121/1.1901928.



Courtesy of the MIT Electrical Engineering and Computer Science Department

Solution

$$\frac{p}{p_\infty} = \frac{32}{3} \frac{\mu\omega}{c_\infty^2 \rho_\infty} \left( \frac{\gamma}{\gamma+1} \right) \sum_{n=1}^{n=\infty} \frac{\sin n \left( \omega t - \omega x / c_\infty \right)}{\sinh n \left[ \log \left[ \frac{16\mu\omega}{3\rho_\infty (\gamma+1) c_\infty^2 K_{1,1}} \right] + \frac{2x\mu\omega^2}{3c_\infty^3 \rho_\infty} \right]}$$

# Guido Fubini-Ghiron

Governing Equation (Conjecture)

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial \xi}{\partial t}\frac{\partial^2 \xi}{\partial x \partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

- Continuous non-conservative one-dimensional finite amplitude waves
- $\xi$ is particle displacement
- Solution via Earnshaw approach
  - Write as binomial series and truncate
  - Convert to Eulerian framework and rewrite as Fourier series

Solution

$$\frac{p}{p_\infty} = \sum_{n=1}^{\infty} \frac{2}{n\sigma} J_n[n\sigma] \sin n\left(\omega t - kx\right)$$

**Fubini-Ghiron, G.**, "Anomalie nella Propagazione di onde Acustiche di Grande Ampiezza," Alta Frequenza, Vol. 4, 1935, pp. 530–581.

Italian, 1879-1943, Professor of Mathematics at Princeton

# David T. Blackstock

Governing Equations (Conjecture)

$$u = g(\phi) \quad \tau = \phi - (\beta c_\infty^{-2}) g(\phi)$$

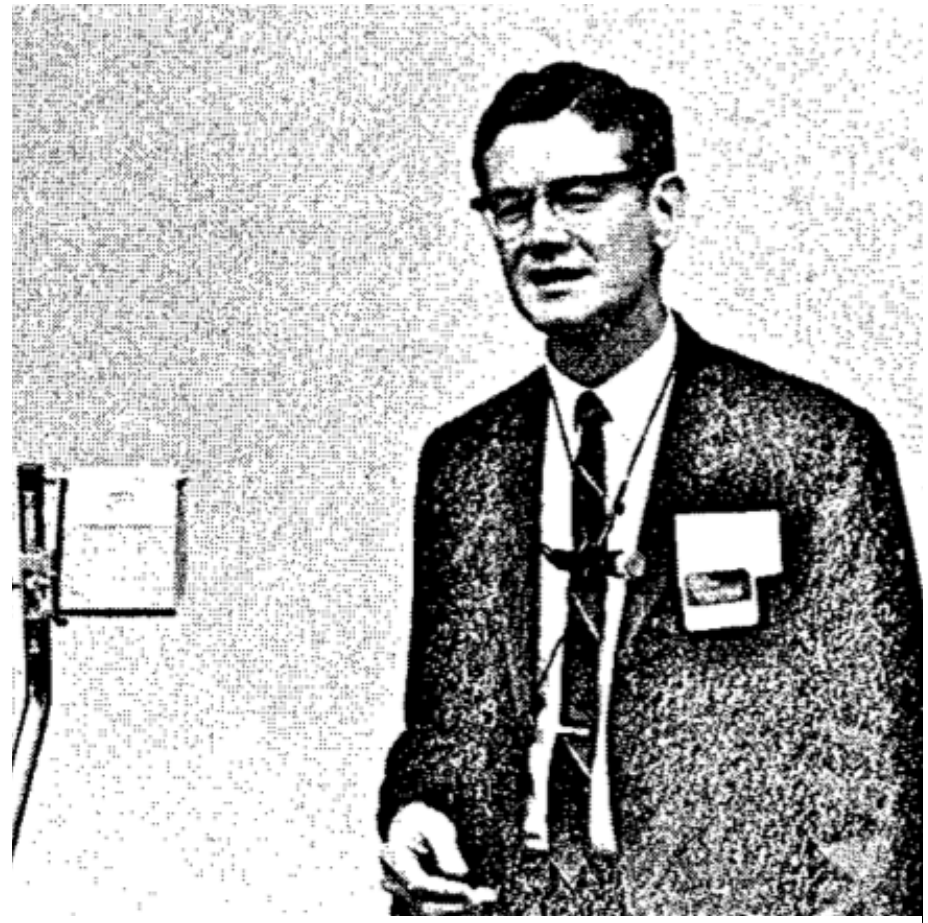
$$\frac{dt'_s}{dx} = -\frac{1}{2}\beta c_\infty^{-2}(u_a + u_b)$$

- Weak shock theory
- $g$ is a function and $\Phi$ is emission time
- Direct solution approach by substitution after eliminating $\Phi$
  - Assume boundary value problem
  - Resultant transcendental equation solved with Fourier series assumption

Solution

$$p(x,t) = p_o \sum_{n=1}^{\infty} B_n \sin\left[n\omega\tau\right]$$

$$B_n = \frac{2}{n(1+\sigma)} + \frac{2}{n\pi\sigma}\int_{\Phi_{sh}}^{\pi} \cos\left[n\left(\Phi - \sigma\sin\Phi\right)\right] d\Phi$$



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# M. J. Lighthill

Governing equations are Navier-Stokes

- Exactly rearrange to form a governing equation, the acoustic analogy

$$\frac{\partial^2 \rho}{\partial t^2} - c_\infty^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

- Right hand side is equivalent source
- Left hand side is linear wave operator

**Lighthill, M. J.**, "On Sound Generated Aerodynamically. I. General Theory," Proc. R. Soc. Lond. A., Vol. 211, No. 1107, 1952, pp. 564–587. doi:10.1098/rspa.1952.0060.

One solution loosely based on Ffowcs Williams

$$S(\boldsymbol{x},\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{r_i r_j r'_l r'_m}{c_\infty^4 r^2 r'^2} g\left(\boldsymbol{x},\boldsymbol{y},\omega\right) g^*\left(\boldsymbol{x},\boldsymbol{y}',\omega\right) \frac{\partial^4}{\partial \tau^4} R_{ijlm}(\boldsymbol{y},\boldsymbol{\eta},\tau)$$
$$\times \exp\left[-i\omega\left(\tau + \frac{r}{c_\infty} - \frac{r'}{c_\infty}\right)\right] d\tau d\boldsymbol{\eta} d\boldsymbol{y}$$

English (born Paris), 1924-1998, Lucasian Professor at Cambridge

# David G. Crighton

Governing equations is Navier-Stokes

- Assume
  - *u is* summation of a gradient and cross-product, eliminate high order terms, flow is irrotational
  - Solutions are set of symmetry
  - *Kr* >> 1, `linear wavenumber'

Governing Equation (non-gen. Burgers')

$$\frac{\partial u}{\partial t} + c_\infty \frac{\partial u}{\partial r} + \frac{\gamma + 1}{2} u \frac{\partial u}{\partial r} + \frac{j c_\infty u}{2r} = \frac{\delta}{2} \frac{\partial^2 u}{\partial r^2}$$

- Spherical, cylindrical, and planar nonlinear wave propagation

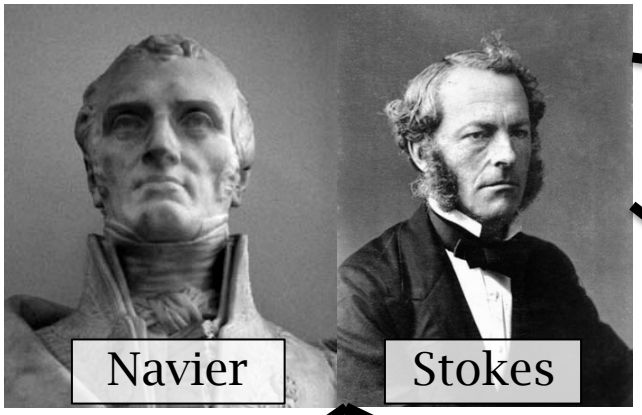
**Crighton, D. G.**, "Model Equations of Nonlinear Acoustics," Annual Review of Fluid Mechanics, Vol. 11, 1979, pp. 11–33. doi:10.1146/annurev.fl.11.010179.000303.

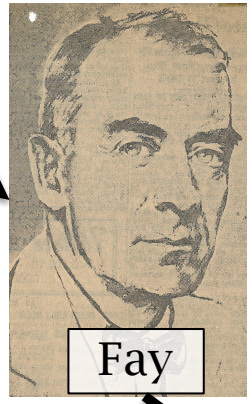
English, 1942-2000, Professor of Applied Mathematics Cambridge
*Also Opera lover – so am I! :)*

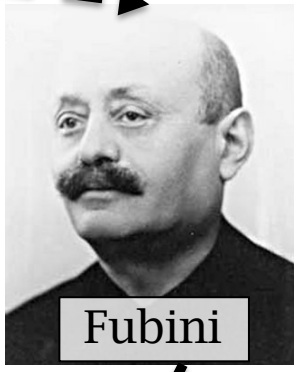
# Mathematical Relationships

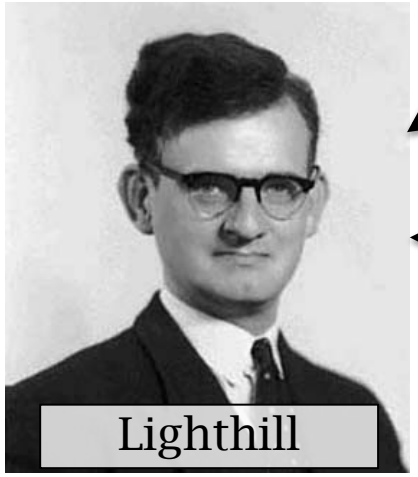
Known Derivation and/or Known Solution →

Newly Developed Derivation and/or Solution - - - →




Presentation Focus



Miller NASA TM shows the mathematical connections and solutions of ALL relations!

# The Navier-Stokes Equations and the Acoustic Analogy

Governing equations are Navier-Stokes.
We now think of the Green's function as satisfying

$$\rho(\boldsymbol{x},t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} d\tau d\boldsymbol{y}$$

We can show using the cross-spectral acoustic analogy

$$S(\boldsymbol{x},\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{r_i r_j r_l' r_m'}{c_\infty^4 r^2 r'^2} g\left(\boldsymbol{x},\boldsymbol{y},\omega\right) g^*\left(\boldsymbol{x},\boldsymbol{y}',\omega\right) \frac{\partial^4}{\partial \tau^4} R_{ijlm}(\boldsymbol{y},\boldsymbol{\eta},\tau)$$
$$\times \exp\left[-i\omega\left(\tau + \frac{r}{c_\infty} - \frac{r'}{c_\infty}\right)\right] d\tau d\boldsymbol{\eta} d\boldsymbol{y}$$

A statistical source model for sound generation (altered from Miller) is

$$\frac{\partial^4}{\partial \tau^4} R_{ijlm}(\boldsymbol{y},\boldsymbol{\eta},\tau) = \frac{4 A_{ijlm} \overline{u}^4}{\pi^{1/2} l_s^8} \left(3 l_s^4 - 12 l_s^2 (\xi - \overline{u}\tau)^2 + 4(\xi - \overline{u}\tau)^4\right)$$
$$\times \exp\left[\frac{-|\xi|}{\overline{u}\tau_s}\right] \exp\left[\frac{-(\xi - \overline{u}\tau)^2}{l_s^2}\right] \exp\left[\frac{-\eta^2}{l_{sy}^2}\right] \exp\left[\frac{-\zeta^2}{l_{sz}^2}\right]$$

**Miller, S. A. E.**, "Prediction of Near-Field Jet Cross Spectra," AIAA Journal, 2015. doi:10.2514/1.J053614.

# The Navier-Stokes Equations and the Acoustic Analogy

Using the source model, assuming that the observer is in the far-field, simplifying, and carefully rearranging yields

$$S(\boldsymbol{x},\omega) = \frac{\pi\omega^4}{c_\infty^4} g\left(\boldsymbol{x},\omega\right) g^*\left(\boldsymbol{x},\omega\right) \int_{-\infty}^{\infty} A_{ijlm} \frac{r_i r_j r_l r_m}{r^4} \frac{l_s l_{sy} l_{sz}}{\overline{u}} \exp\left[\frac{-l_s^2\omega^2}{4\overline{u}^2}\right]$$

$$\times \int_{-\infty}^{\infty} \exp\left[\frac{-i\xi\omega}{\overline{u}}\right] \exp\left[\frac{-|\xi|}{\overline{u}\tau_s}\right] d\xi dy_1$$

Green's function

Source Spectrum

Selective far-field assumption

- Source remains a volumetric integral
- Propagation approximated from a point within source volume

Need to find what *gg** is to capture nonlinear propagation effects

# The Navier-Stokes Equations and a Burgers' Equation

The Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \qquad \text{and} \qquad \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j}$$

Following Crighton then finding a more compact form governing pressure

$$\frac{\partial p}{\partial x} + m\frac{p}{r} - \epsilon p \frac{\partial p}{\partial \tau} = \frac{\delta}{2c_\infty^3}\frac{\partial^2 p}{\partial \tau^2}$$

Select analytical solutions exist – eg: Blackstock, Fay, and Fubini

Seek a numerical solution in the frequency domain (as shown by Saxena)

$$\frac{\partial \tilde{p}}{\partial r} + m\frac{\tilde{p}}{r} + (\alpha + i\beta)\,\tilde{p} = \frac{i\omega\epsilon}{2}\tilde{q}$$

Pseudo-spectral numerical method marches solution in space from prescribed boundary condition (same BC as Blackstock)

# The Connection Between the Acoustic Analogy and Generalized Burgers' Equation

Conjecture: Given the solution of

$$\frac{\partial \tilde{p}}{\partial r} + m\frac{\tilde{p}}{r} + (\alpha + i\beta)\,\tilde{p} = \frac{i\omega\epsilon}{2}\tilde{q}$$

subject to the boundary condition of a source spectrum of the acoustic analogy and $x >> D$ then

$$g(\boldsymbol{x},\omega)g^*(\boldsymbol{x},\omega) \approx \tilde{p}(\boldsymbol{x},\omega)\tilde{p}^*(\boldsymbol{x},\omega)$$

within the acoustic analogy. As $\lim \tilde{p} \to \epsilon$

$$g(\boldsymbol{x},\omega)g^*(\boldsymbol{x},\omega) = \tilde{p}(\boldsymbol{x},\omega)\tilde{p}^*(\boldsymbol{x},\omega)$$

for the traditional approach only.

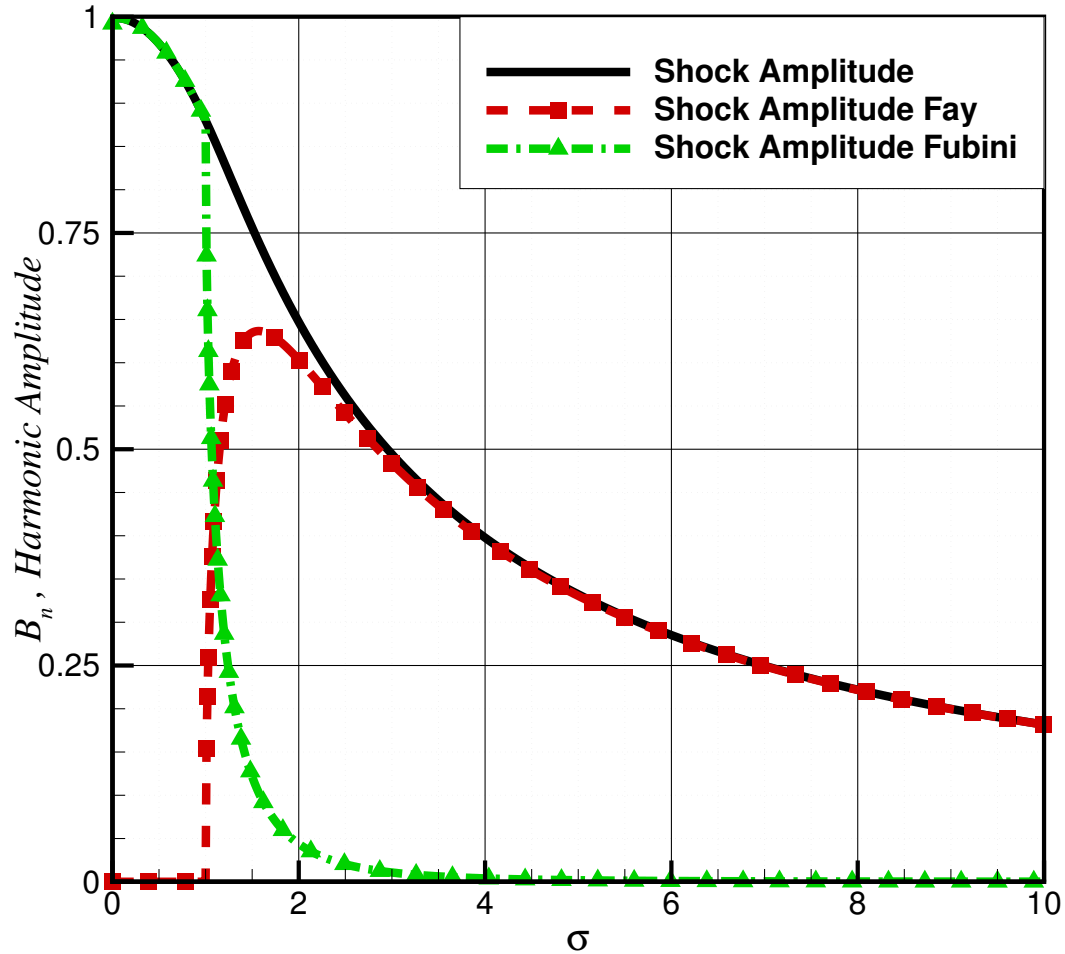
# Acoustic Analogy and Burgers' Equation

- Approximation of $gg^*$ is obtained from solution of generalized Burgers' equation
- Boundary condition (at $r = 0$) of generalized Burgers' equation is broadband source spectrum
- Source spectrum at low intensities results in predictions that are equivalent to those produced by traditional acoustic analogies
- Source spectrum at high intensity causes nonlinear terms within generalized Burgers' equation to be dominant
- Characteristics of nonlinear propagation are apparent in predicted jet mixing noise spectrum.

# Results

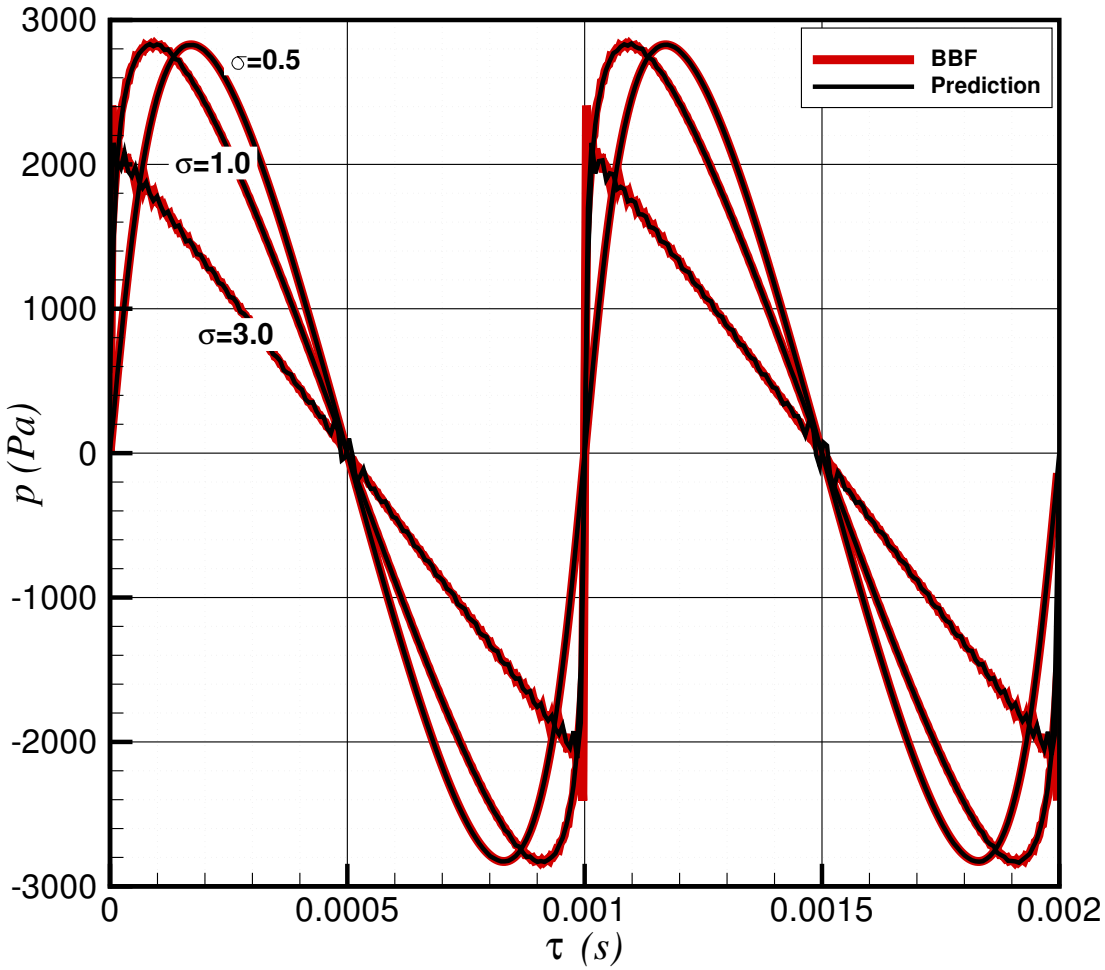
# Examination of Blackstock, Fay, and Fubini

- Almost exact solutions of generalized Burgers' equation
- Source planar sine wave at 160dB and 1000Hz
- Regions of validity



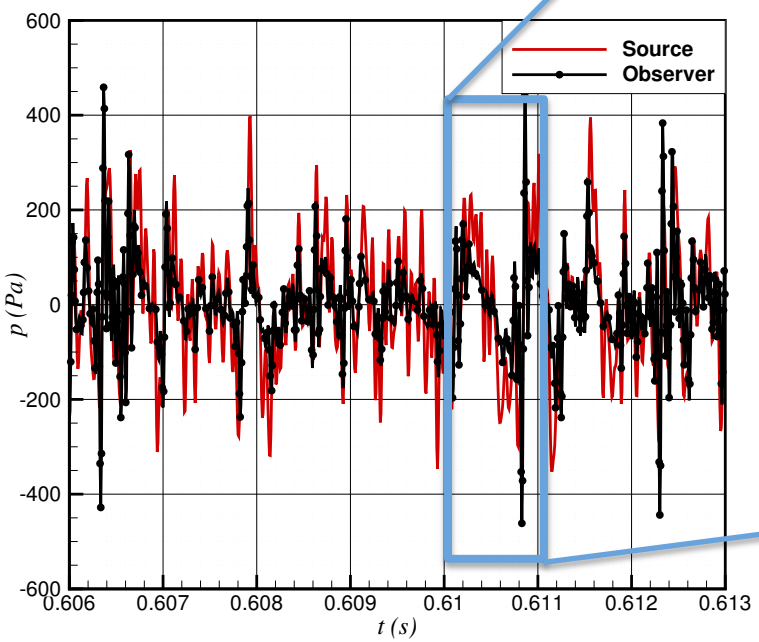
# Numerical Solver of Generalized Burgers' Equation and Blackstock Bridging Function

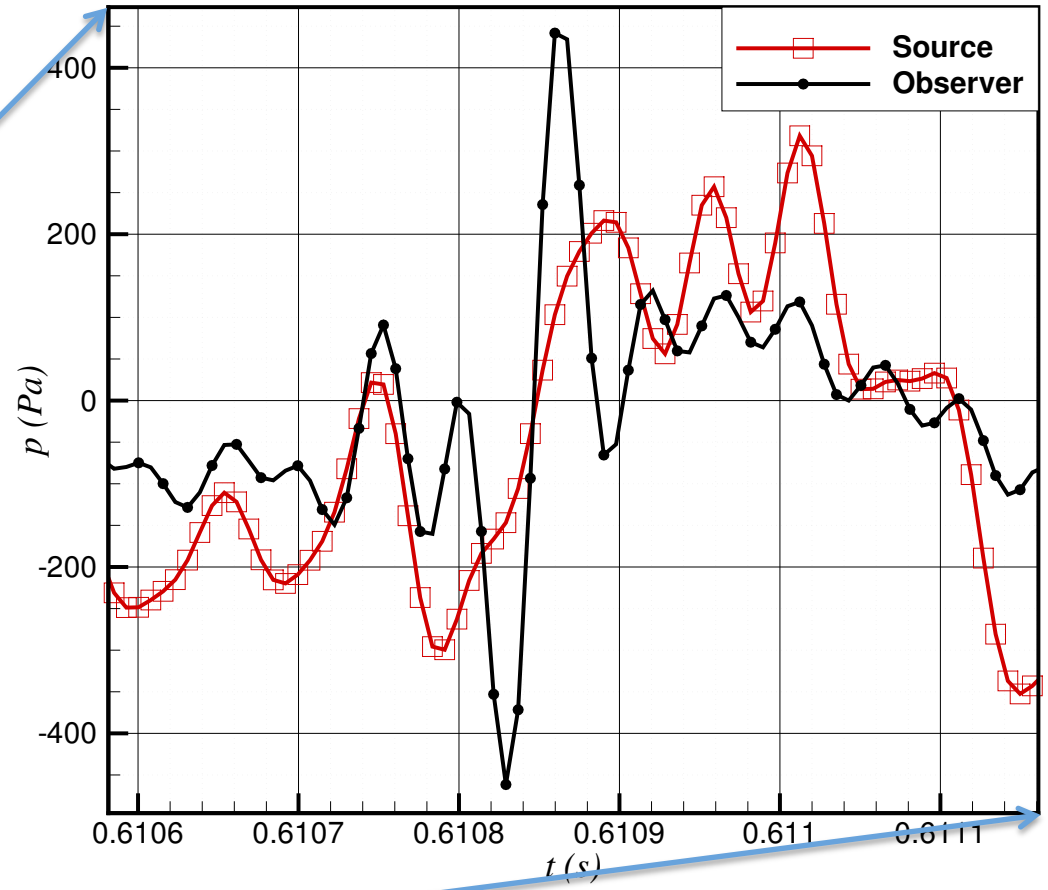
- Comparison at three observer positions
- Numerical solver agrees with analytic result
- Source planar sine wave at 160dB and 1000Hz
- Gibb's phenomenon present



# Propagation of a Broadband Signal

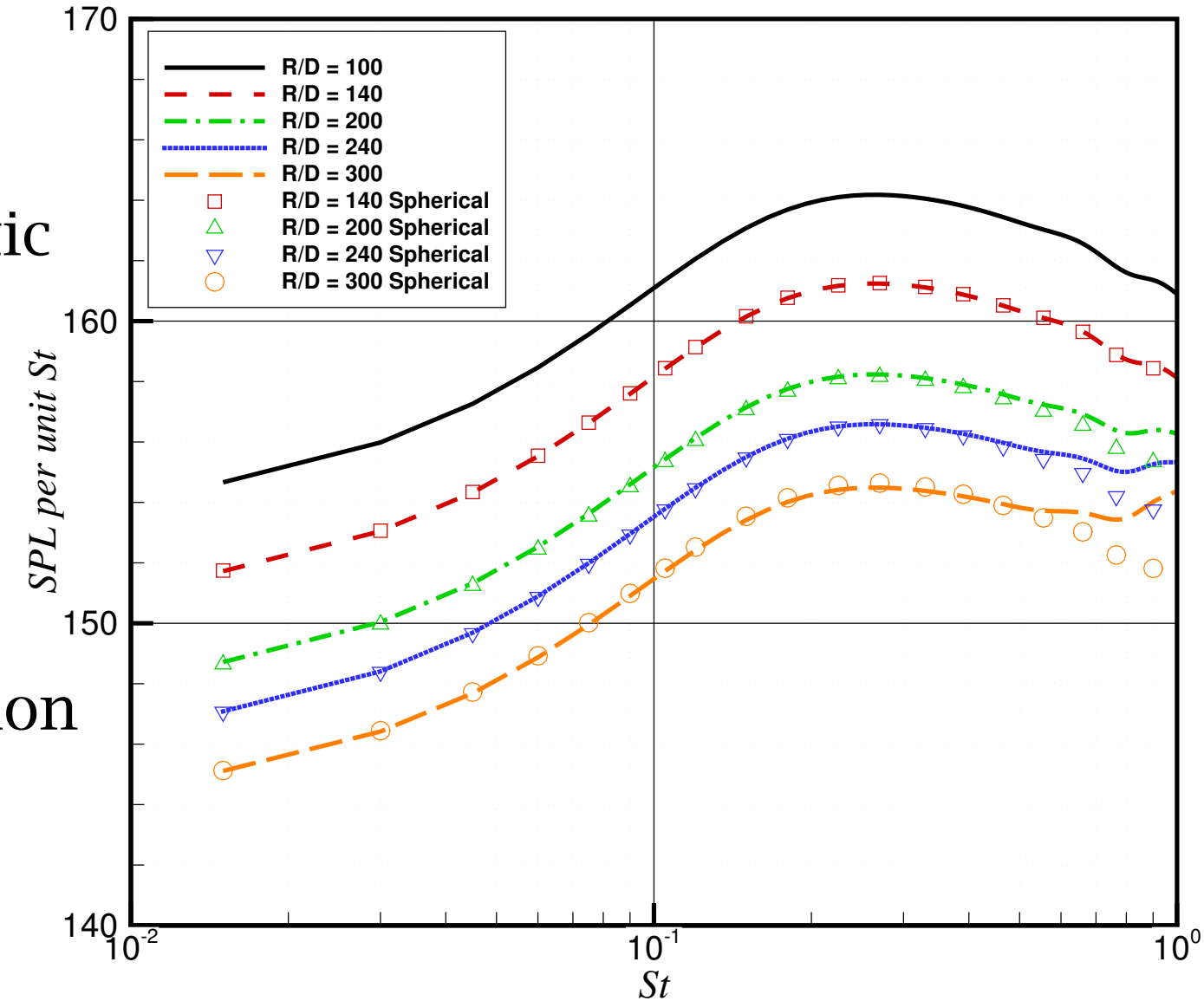
- Shocked observer waveform due to wave coalescence
- Discontinuities not present in source signal
- Not observed in linear acoustics
- Jet $M_j = 1.86$
- Jet TTR = 3.20
- Sideline direction



# Example Jet Noise Prediction Directly Incorporating Nonlinear Propagation

- Unified Acoustic Analogy with Nonlinear Propagation
- Jet $M_j = 1.86$
- Jet TTR = 3.20
- Sideline direction



# Summary and Conclusion

- Showed connection between Navier-Stokes equations, generalized Burgers' equation (sound propagation), and Acoustic Analogy (sound source)
- Nonlinear propagation taken into account directly from source to observer
- A single equation contains sound source and nonlinear propagation from turbulence
- Evaluated select equations to demonstrate relevant physics

# Questions

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