Toward a Nonlinear Acoustic Analogy: Turbulence as a Source of Sound and Nonlinear Propagation

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• Many previous curious researchers
Introduction
Turbulence and Nonlinear Propagation

- Aerospace vehicles produce turbulence
- Sound propagates nonlinearly if turbulence is highly intense
- Intense noise is harmful to the vehicle and environment

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Rocket noise has nonlinear propagation effects NASA SMAT Test (NASA.gov)

Airbreathing engine jet noise propagation effects NASA CST Test (NASA.gov)
Turbulence and Nonlinear Propagation

- Understand different mathematical models of sound generation and propagation
- Relate the governing equations to sound generation and propagation
- Show a mathematical connection between sound generation (acoustic analogy) and sound propagation (Burgers’ equation)

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Mathematical Models
Claude-Louis Navier and George Gabriel Stokes

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \]
\[ \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} \]
\[ \frac{\partial \rho e_o}{\partial t} + \frac{\partial \rho u_j e_o}{\partial x_j} = -\frac{\partial u_j p}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \frac{\partial u_i \tau_{ij}}{\partial x_j} \]

Claude-Louis Marie Henri Navier
- 1735-1836
- French
- Professor at École Nationale des Ponts et Chaussées
- Known for elasticity and structural engineering

Sir George Stokes
- 1819-1903
- Irish
- Lucasian Professor
- Fluids, Optics, Chemistry
- Politics and Theology

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Governing Equation (Conjecture)

\[
c_\infty^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial x} \gamma^+ \left[ \frac{\partial^2 y}{\partial t^2} - \frac{4\mu}{3\rho_\infty} \frac{\partial}{\partial t} \left( \frac{\partial^2 y}{\partial x^2} \right) \right]
\]

- Governs shocked one-dimensional finite amplitude waves
- \(y\) is particle displacement
- Solution via assumptions
  - Periodic
  - \(dy/dx\) is Fourier series
  - Substitute and solve Fourier series


Solution

\[
\frac{p}{p_\infty} = \frac{32}{3} \frac{\mu \omega}{c_\infty^2 \rho_\infty} \left( \frac{\gamma}{\gamma + 1} \right) \sum_{n=1}^{\infty} \sinh n \left[ \log \left( \frac{16\omega}{3\rho_\infty (\gamma + 1)c_\infty^2 K_{1,1}} \right) + \frac{2x \mu \omega^2}{3c_\infty^3 \rho_\infty} \right]
\]
Guido Fubini-Ghiron

Governing Equation (Conjecture)

\[
\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial \xi}{\partial t} \frac{\partial^2 \xi}{\partial x \partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x}
\]

- Continuous non-conservative one-dimensional finite amplitude waves
- \( \xi \) is particle displacement
- Solution via Earnshaw approach
  - Write as binomial series and truncate
  - Convert to Eulerian framework and rewrite as Fourier series

Solution

\[
\frac{p}{p_\infty} = \sum_{n=1}^{\infty} \frac{2}{n\sigma} J_n[n\sigma] \sin n(\omega t - kx)
\]

Governing Equations (Conjecture)

\[ u = g(\phi) \quad \tau = \phi - (\beta c_\infty^{-2})g(\phi) \]
\[ \frac{dt'}{dx} = -\frac{1}{2} \beta c_\infty^{-2}(u_a + u_b) \]

- Weak shock theory
- \( g \) is a function and \( \phi \) is emission time
- Direct solution approach by substitution after eliminating \( \phi \)
  - Assume boundary value problem
  - Resultant transcendental equation solved with Fourier series assumption

Solution

\[ p(x, t) = p_o \sum_{n=1}^{\infty} B_n \sin [n\omega \tau] \]
\[ B_n = \frac{2}{n(1 + \sigma)} + \frac{2}{n\pi\sigma} \int_{\Phi_{sh}}^{\pi} \cos [n (\Phi - \sigma \sin \Phi)] d\Phi \]

Governing equations are Navier-Stokes
• Exactly rearrange to form a governing equation, the acoustic analogy
\[ \frac{\partial^2 \rho}{\partial t^2} - c_\infty^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \]
• Right hand side is equivalent source
• Left hand side is linear wave operator


One solution loosely based on Ffowcs Williams

\[
S(x, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{r_i r_j r_l r_m}{c_\infty^4} \frac{g(x, y, \omega)}{r^2 r'^2} g^*(x, y', \omega) \frac{\partial^4}{\partial \tau^4} R_{ijlm}(y, \eta, \tau) \times \exp \left[ -i \omega \left( \tau + \frac{r}{c_\infty} - \frac{r'}{c_\infty} \right) \right] \, d\tau \, d\eta \, dy
\]
Governing equations is Navier-Stokes

- Assume
  - $u$ is summation of a gradient and cross-product, eliminate high order terms, flow is irrotational
  - Solutions are set of symmetry
  - $Kr >> 1$, `linear wavenumber’

Governing Equation (non-gen. Burgers’)

$$\frac{\partial u}{\partial t} + c_\infty \frac{\partial u}{\partial r} + \frac{\gamma + 1}{2} u \frac{\partial u}{\partial r} + \frac{j c_\infty u}{2r} = \delta \frac{\partial^2 u}{\partial r^2}$$

- Spherical, cylindrical, and planar nonlinear wave propagation


English, 1942-2000, Professor of Applied Mathematics Cambridge
Also Opera lover – so am I! :)}
Mathematical Relationships

Known Derivation and/or Known Solution

Newly Developed Derivation and/or Solution

Navier

Stokes

Fubini

Fay

Lighthill

Crighton

Presentation Focus

Blackstock

Miller NASA TM shows the mathematical connections and solutions of ALL relations!

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The Navier-Stokes Equations and the Acoustic Analogy

Governing equations are Navier-Stokes. We now think of the Green’s function as satisfying

$$\rho(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} d\tau dy$$

We can show using the cross-spectral acoustic analogy

$$S(x, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{r_i r_j r'_l r'_m}{c^4 \tau^2 \tau'2} g(x, y, \omega) g^*(x, y', \omega) \frac{\partial^4}{\partial \tau^4} R_{ijlm}(y, \eta, \tau)$$

$$\times \exp \left[ -i\omega \left( \tau + \frac{r}{c_\infty} - \frac{r'}{c_\infty} \right) \right] d\tau d\eta dy$$

A statistical source model for sound generation (altered from Miller) is

$$\frac{\partial^4}{\partial \tau^4} R_{ijlm}(y, \eta, \tau) = \frac{4A_{ijlm} u_4}{\pi^{1/2} l_s^8} \left( 3l_s^4 - 12l_s^2 (\xi - u\tau)^2 + 4(\xi - u\tau)^4 \right)$$

$$\times \exp \left[-\frac{|\xi|}{u\tau_s} \right] \exp \left[-\frac{(\xi - u\tau)^2}{l_s^2} \right] \exp \left[-\frac{\eta^2}{l_{sy}^2} \right] \exp \left[-\frac{\zeta^2}{l_{sz}^2} \right]$$

The Navier-Stokes Equations and the Acoustic Analogy

Using the source model, assuming that the observer is in the far-field, simplifying, and carefully rearranging yields

\[ S(\mathbf{x}, \omega) = \frac{\pi \omega^4}{c_\infty^4} g(\mathbf{x}, \omega) g^*(\mathbf{x}, \omega) \int_{-\infty}^{\infty} A_{ijlm} \frac{r_i r_j r_l r_m}{r^4} \frac{l_s l_s y l_s z}{u} \exp \left[ \frac{-l_s^2 \omega^2}{4u^2} \right] \]

\[ \times \int_{-\infty}^{\infty} \exp \left[ -i \xi \omega \right] \exp \left[ -\frac{|\xi|}{u \tau_s} \right] d\xi dy_1 \]

Selective far-field assumption
- Source remains a volumetric integral
- Propagation approximated from a point within source volume

Need to find what \( gg^* \) is to capture nonlinear propagation effects
The Navier-Stokes Equations

and a Burgers’ Equation

The Navier-Stokes equations

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \]

and

\[ \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} \]

Following Crighton then finding a more compact form governing pressure

\[ \frac{\partial p}{\partial x} + m \frac{p}{r} - \epsilon p \frac{\partial p}{\partial \tau} = \frac{\delta}{2 c_\infty^3} \frac{\partial^2 p}{\partial \tau^2} \]

Select analytical solutions exist – eg: Blackstock, Fay, and Fubini

Seek a numerical solution in the frequency domain (as shown by Saxena)

\[ \frac{\partial \tilde{p}}{\partial r} + m \frac{\tilde{p}}{r} + (\alpha + i \beta) \tilde{p} = \frac{i \omega \epsilon}{2} \tilde{q} \]

Pseudo-spectral numerical method marches solution in space from prescribed boundary condition (same BC as Blackstock)
The Connection Between the Acoustic Analogy and Generalized Burgers’ Equation

Conjecture: Given the solution of

$$\frac{\partial \tilde{p}}{\partial r} + m \frac{\tilde{p}}{r} + (\alpha + i \beta) \tilde{p} = \frac{i \omega \epsilon}{2} \tilde{q}$$

subject to the boundary condition of a source spectrum of the acoustic analogy and $x >> D$ then

$$g(x, \omega)g^*(x, \omega) \approx \tilde{p}(x, \omega)\tilde{p}^*(x, \omega)$$

within the acoustic analogy. As $\lim \tilde{p} \rightarrow \epsilon$

$$g(x, \omega)g^*(x, \omega) = \tilde{p}(x, \omega)\tilde{p}^*(x, \omega)$$

for the traditional approach only.
Acoustic Analogy and Burgers’ Equation

• Approximation of $gg^*$ is obtained from solution of generalized Burgers’ equation
• Boundary condition (at $r = 0$) of generalized Burgers’ equation is broadband source spectrum
• Source spectrum at low intensities results in predictions that are equivalent to those produced by traditional acoustic analogies
• Source spectrum at high intensity causes nonlinear terms within generalized Burgers’ equation to be dominant
• Characteristics of nonlinear propagation are apparent in predicted jet mixing noise spectrum.
Results
• Almost exact solutions of generalized Burgers’ equation
• Source planar sine wave at 160dB and 1000Hz
• Regions of validity
Numerical Solver of Generalized Burgers’ Equation and Blackstock Bridging Function

- Comparison at three observer positions
- Numerical solver agrees with analytic result
- Source planar sine wave at 160dB and 1000Hz
- Gibb’s phenomenon present
Propagation of a Broadband Signal

- Shocked observer waveform due to wave coalescence
- Discontinuities not present in source signal
- Not observed in linear acoustics
- Jet $M_j = 1.86$
- Jet TTR = 3.20
- Sideline direction

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Example Jet Noise Prediction
Directly Incorporating Nonlinear Propagation

- Unified Acoustic Analogy with Nonlinear Propagation
- Jet $M_j = 1.86$
- Jet TTR = 3.20
- Sideline direction
Summary and Conclusion

• Showed connection between Navier-Stokes equations, generalized Burgers’ equation (sound propagation), and Acoustic Analogy (sound source)

• Nonlinear propagation taken into account directly from source to observer

• A single equation contains sound source and nonlinear propagation from turbulence

• Evaluated select equations to demonstrate relevant physics
Questions
References

References

References


