Assessment of an Analytical Prediction Technique for the Shock Wave Attached to a Cone at an Angle of Attack

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Importance & Objectives

Importance:

- Useful for rapid approximate calculations for design
- Allow for an increased understanding of the physics of high-speed flows

Objective:

- Find fast accurate analytical solutions for high-speed inviscid flow-fields, eventually relaxing assumptions to predict flow-fields for more complex geometries
- Assessment of current methods is the first step

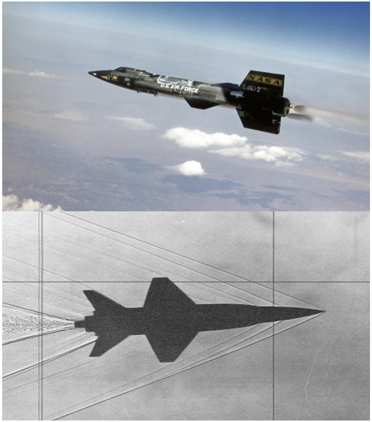

Figure 1: X-15 hypersonic aircraft (top) [1] and shadowgraph of wind tunnel model (bottom) [2].

Taylor-Maccoll (1933)

- By definition of conical flow, flow-variables are functions of polar coordinate (θ) only
- System of ODEs representing an exact solution to inviscid axisymmetric irrotational flow [3]:

$$\frac{\gamma - 1}{2}\left[1 - u_r^2 - \left(\frac{du_r}{d\theta}\right)^2\right]\left[2u_r + \cot(\theta)\frac{du_r}{d\theta} + \frac{d^2u_r}{d\theta^2}\right] - \frac{du_r}{d\theta}\left[u_r\frac{du_r}{d\theta} + \frac{du_r}{d\theta}\frac{d^2u_r}{d\theta^2}\right] = 0$$

$$u_\theta = \frac{du_r}{d\theta}$$

Stone (1948)

- Flow variables are expressed in a power series in α (angle of attack) [4]. Neglecting higher terms gives:

$$
\begin{aligned}
u &= u_1 + \alpha u_2 \cos\phi \\
v &= v_1 + \alpha v_2 \cos\phi \\
w &= \alpha w_2 \sin\phi \\
P &= P_1 + \alpha P_2 \cos\phi \\
\rho &= \rho_1 + \alpha \rho_2 \cos\phi
\end{aligned}
$$

- The following differential equation is solved, where coefficients A, B, and C are functions of the Taylor-Maccoll solution and f is a function of $(P_1,\ P_2,\ ,\rho_1,\ \rho_2)$:

$$\frac{1}{f}\left(\frac{d^2 u_2}{d\theta^2}\right) + \frac{A}{f}\left(\frac{du_2}{d\theta}\right) + \frac{B}{f}u_2 + C = 0$$

Savin (1951)

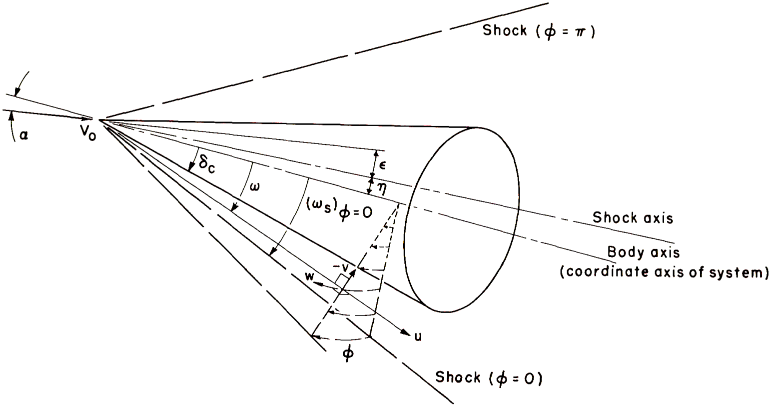

Figure 2: Geometry and coordinate system [5].

Savin (1951)

- Governing equation in the plane of symmetry (irrotational):

$$\frac{\gamma-1}{2}\left[\left(\frac{\hat{V}}{V}\right)^2-1\right]\left[\cot(\omega-\delta)\left(1+\frac{\partial\delta}{\partial\omega}\right)-\cot\omega \right.$$
$$\left. +\tan(\omega-\delta)\frac{\partial\delta}{\partial\omega}+\frac{\csc(\omega-\delta)}{V\sin\omega}\frac{\partial w}{\partial\phi}\right]-\tan(\omega-\delta)\frac{\partial\delta}{\partial\omega}=0 \tag{1}$$

- Assume a conical shock with circular cross-sections:

$$\omega_s=(\omega_s)_{\phi=0}+\eta(1-\cos\phi) \tag{2}$$

- This leads to:

$$(\omega_s)_{\phi=\pi}-(\omega_s)_{\phi=0}=2\eta \tag{3}$$

- An expression for δ in the plane of symmetry is obtained from governing equation by assuming either $\omega-\delta << 1$ or $\delta << 1$
- Solve a system of nonlinear *algebraic* equations for ω_s until (3) is satisfied

Savin's Equations

- Assume $\omega - \delta << 1$ or $\delta << 1$ only no solution is available with first assumption
- Equation (1) becomes analytically integrable to obtain an equation governs the entire plane of symmetry

$$\delta_s = \delta_s(M_s, \delta_c, V_0, \epsilon) \tag{4}$$

- Mach number after shock (M_s):

$$M_s = \sqrt{\frac{M_0^4 \cdot (\gamma+1)^2 \sin^2(\omega_s \pm \alpha) - 4\left(M_0^2 \sin^2(\omega_s \pm \alpha) - 1\right)\left(M_0^2 \gamma \sin^2(\omega_s \pm \alpha) + 1\right)}{\left(M_0^2(\gamma-1)\sin^2(\omega_s \pm \alpha) + 2\right)\left(2M_0^2 \gamma \sin^2(\omega_s \pm \alpha) - \gamma + 1\right)}} \tag{5}$$

- Deflection angle after shock (δ_s):

$$\delta_s \pm \alpha = \tan^{-1}\left(\frac{\cot(\omega_s \pm \alpha)\left(M_0^2 \sin^2(\omega_s \pm \alpha) - 1\right)}{\frac{\gamma+1}{2} M_0^2 - M_0^2 \sin^2(\omega_s \pm \alpha) + 1}\right) \tag{6}$$

Comparison of Methods

	Taylor-Maccoll	Stone	Savin
Assumptions:	Inviscid Irrotational Axisymmetric	Inviscid Velocity components vary linearly with α	Inviscid $\omega - \delta << 1$ or $\delta << 1$
Range of Validity:	α=0°	Small α	$M_0\delta_c \geq 1$
Computation:	Numerical integration	Numerical integration	Fully analytical, Numerical root-finding used to solve algebraic system

Savin Versus Taylor-Maccoll ($\alpha = 0^\circ$)

- Better agreement with higher Mach numbers and cone angles

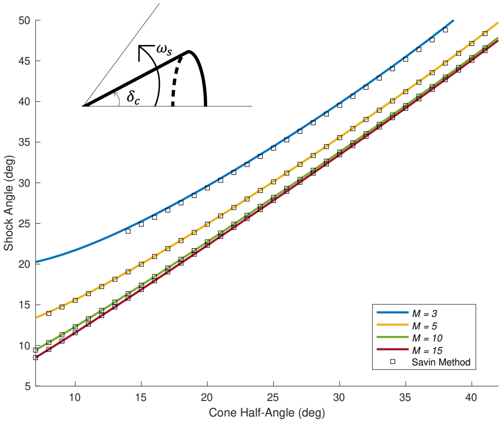

Figure 3: $\delta_c - \omega_s - M$ relation.

Savin Versus Stone First-Order

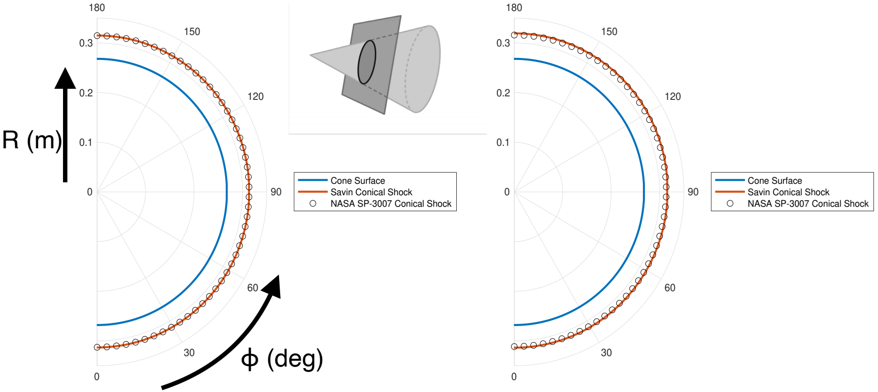

Figure 4: Cross-stream view 1 m downstream of vertex of shock in Mach 10 flow past 15° half-angle cone at 3° angle of attack (left) and 5° angle of attack (right).

Savin Versus Stone First-Order

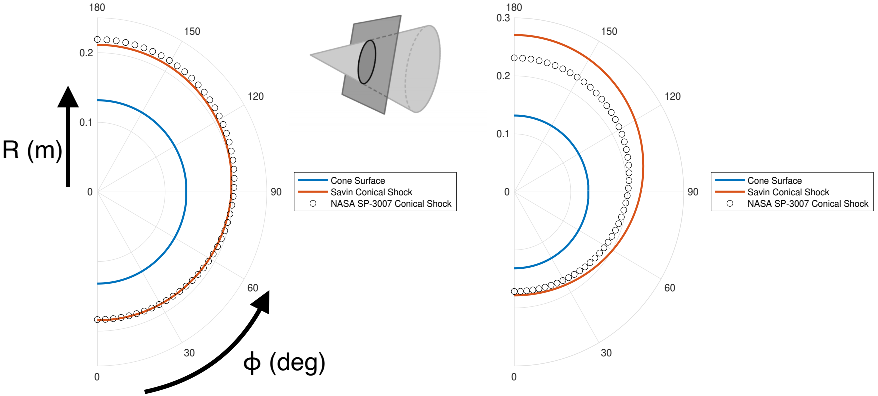

Figure 5: Cross-stream view 1 m downstream of vertex of shock in Mach 7 flow past 7.5° half-angle cone at 3° angle of attack (left) and 5° angle of attack (right).

Predictions

- Simplicity of Savin's method allows for predictions like the following to be made in seconds with shock-expansion theory:

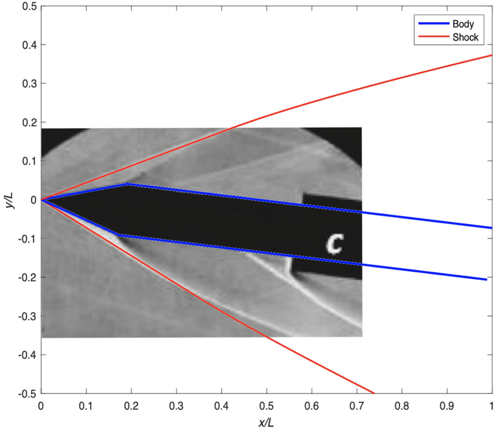

Figure 6: Near-field shock comparison with experiment for Mach 3 flow past 20° cone at 8° angle of attack [7].

Conclusions

- Savin's method has good agreement with other well-known methods
  - Has a maximum error of less than half a degree from the Taylor-Maccoll solution
  - Produces similar results to Stone's theory for hypersonic similarity parameters greater than 1

- Future work: make similar assumptions or relax current assumptions to extend current methods to obtain shock shapes and flow-fields for more complex geometries.

Thank You!
Questions?

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[5] Raymond C. Savin, "Application of the Generalized Shock- Expansion Method to Inclined Bodies of Revolution Traveling at High Supersonic Airspeeds," NACA TN 3349, 1955.

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[7] Simonenko, M. M., Guvernyuk, S. V, and Kuzmin, A. G.. "On the Supersonic Flow over an Axisymmetric Step at an Angle of Attack." AIP Conference Proceedings, 2018, https://doi.org/10.1063/1.5065117.

Extra Slides - Savin's Equations

$$\delta_s = \ln\left[1 - M_s\left(\omega_s - \delta_c\right)(\sigma + 2)\right]\left[\left(M_s\left(M_s \tan\left(\delta_c\right)(\sigma + 2) - \sigma - 1\right) + \tan\left(\delta_c\right)\right)\sin^2\left(\frac{1}{M_s\ (\sigma + 2)} + \delta_c\right) + M_s \sin^2\left(\delta_c\right)\sigma\left(M_s \tan\left(\delta_c\right)(\sigma + 2) + 1\right)\right]\cdot$$

$$\left[2M_s^2\ (\sigma + 2)\left(M_s \tan\left(\delta_c\right)(\sigma + 2) + 1\right)\sin^2\left(\frac{1}{M_s\ (\sigma+2)} + \delta_c\right)\right]^{-1} +$$

$$\sin^2\left(\delta_c\right)\left(\cot\left(\omega_s\right) - \cot\left(\delta_c\right)\right)\sigma\left[1 - \frac{\sin^2\left(\delta_c\right)\left(\csc^2\left(\frac{1}{M_s\ (\sigma+2)} + \delta_c\right) + \csc^2\left(\frac{1}{M_s\ (\sigma+2)} - \delta_c\right)\right)}{2}\right] +$$

$$\frac{\ln\left[M_s\ \left(\omega_s - \delta_c\right)(\sigma + 2) + 1\right]\left[\left(M_s\ \left(M_s \tan\left(\delta_c\right)(\sigma + 2) + \sigma + 1\right) + \tan\left(\delta_c\right)\right)\sin^2\left(\frac{1}{M_s\ (\sigma+2)} - \delta_c\right) - M_s \sin^2\left(\delta_c\right)\ \sigma\ \left(1 - M_s \tan\left(\delta_c\right)(\sigma + 2)\right)\right]}{2M_s^2\ (\sigma + 2)\left(1 - M_s \tan\left(\delta_c\right)(\sigma + 2)\right)\sin^2\left(\frac{1}{M_s\ (\sigma+2)} - \delta_c\right)} +$$

$$\frac{\sec^2\left(\delta_c\right)\tan\left(\delta_c\right)\ln\left(\cot\left(\delta_c\right)\left(\omega_s - \delta_c\right) + 1\right)(\sigma + 2)}{M_s^2\ \tan^2\left(\delta_c\right)(\sigma + 2)^2 - 1} + \delta_c \qquad (7)$$

where σ is given by

$$\sigma = \pm\frac{V_0}{V}\epsilon\frac{\sin\,\omega_s}{\sin^2\,\delta_c}$$

Extra Slides - Savin's Equations

$$\delta_s = \delta_c \left(\frac{\frac{2M_s^2\left(\sin^2(\delta_c)-\sin^2(\omega_s)\right)(\sigma+1)}{\sqrt{\left(M_s^2\sin^2(\delta_c)\,\sigma+1\right)^2+4M_s^2\sin^2(\delta_c)}-M_s^2\sin^2(\delta_c)\,\sigma+1}+1}{-\frac{2M_s^2\left(\sin^2(\delta_c)-\sin^2(\omega_s)\right)(\sigma+1)}{\sqrt{\left(M_s^2\sin^2(\delta_c)\,\sigma+1\right)^2+4M_s^2\sin^2(\delta_c)}-M_s^2\sin^2(\delta_c)\,\sigma+1}+1} \right)^{\frac{\frac{\sigma\left(M_s^2\sin^2(\delta_c)\left(-\sigma^2+2\sigma+4\right)-\sigma-\sin^2(\delta_c)\right)}{M_s^2\sin^2(\delta_c)\left(4-\sigma^2\right)-2\sigma}+1}{2\sqrt{\left(M_s^2\sin^2(\delta_c)\,\sigma+1\right)^2+4M_s^2\sin^2(\delta_c)}}} .$$

$$\left(\frac{\left[2-\sin^2\omega_s\left(1+\sigma\csc^2\delta_c\right)\right]^2}{\left[2-(\sigma+\sin^2\delta_c)\right]^2\left(\frac{\sin^2\omega_s}{\sin\delta_c}\left[1+(2+\sigma)M_s^2\sin^2\delta_c-(1+\sigma)M_s^2\sin^2\omega_s\right]-M_s^2\sin^2\delta_c\right)} \right)^{\frac{\sigma(\sigma+\sin^2\delta_c)}{2\sigma-M^2\sin^2\delta_c(4-\sigma^2)}} \tag{8}$$

Extra Slides - Savin Method Applicability