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Research Article

Analytical Equations for Thermoacoustic Instability Sources and Acoustic Radiation from Reacting Turbulence

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We seek to ascertain and understand source terms that drive thermoacoustic instability and acoustic radiation. We present a new theory based on the decomposition of the Navier-Stokes equations coupled with the mass fraction equations. A series of solutions are presented via the method of the vector Green's function. We identify both combustion-combustion and combustion-aerodynamic interaction source terms. Both classical combustion noise theory and classical Rayleigh criterion are recovered from the presently developed more general theory. An analytical spectral prediction method is presented, and the two-point source terms are consistent with Lord Rayleigh's instability model. Particular correlations correspond to the source terms of Lighthill, which represent the noise from turbulence and additional terms for the noise from reacting flow.

1. Introduction

We present analytical source terms for acoustic radiation from turbulent reacting flow. We also present source terms that contribute to thermoacoustic radiation. These source terms are related. Through this approach, we recover the traditional analytical theories of acoustic radiation that are related to the empirically validated unsteady heat release. We also show that the traditional Raleigh criterion is a subset of the current method. Using this approach, we show both time-domain and stochastic closed-form prediction models for acoustic fluctuations. Through formation of the spectral density of acoustic pressure, we find two-point correlations of source statistics that are important and exact for thermoacoustic stability models.

We review traditional aerodynamic noise theory before we review traditional combustion theory, which is present in both reacting and nonreacting flows. Lighthill [1] proposed the acoustic analogy. The acoustic analogy is a rearrangement of the Navier-Stokes equations into a wave equation that is equated with equivalent sources. Lighthill's analysis shows that the noise from fully developed turbulence scales as the u^8 power, where u is a velocity scale of the flow. The importance of Lighthill's analysis in combustion is that it is compatible with traditional theory, which involves a source term of unsteady heat release. However, the approach remains an equivalent source technique, and the Green's function cannot account directly for refraction effects. Lighthill's approach only quantifies the aerodynamic fluctuations, which are highly dependent on $\rho u_i u_j$. The approach also has a number of drawbacks, which include the difficulty of not accounting for refraction effects inherent in acoustic propagation. We use the fundamental idea of the acoustic analogy in the present approach but retain a more complicated form of the vector Green's function to capture refraction effects.

Ffowcs Williams and Hawkings [2] (FWH) developed an acoustic analogy to account for the noise from moving surfaces. The FWH method uses the same approach as Lighthill [1] with two additional fundamental concepts. The first is the use of generalized functions. The second is the introduction of a fictitious simple closed surface within the flow-field. For practical applications and predictions, the integration surface is defined to coincide with a physical surface of a combustion chamber and planes that radiating waves propagate through. Alternatively, the surface can be placed within a space that encompasses all sources with the requirement (that is often not met in practice) that no vorticity impacts the integration surface. The solution of the FWH equation is not trivially derived, and one of the approaches of Farassat [3] is the most amenable for evaluation. Farassat [3] derived "Formulation 1A" to predict pressure from the FWH surface. These formulations are practical for the prediction of aerodynamic noise but do not account explicitly for combustion. However, their source models are still present in turbulent reacting flow and contribute to the radiated noise. Techniques pioneered by FWH and Farassat are excellent and practical tools for prediction of combustion noise but cannot ascertain the thermoacoustic instability sources.

We now turn our attention to combustion noise sources. A comprehensive review of premixed combustion-acoustic wave interaction is presented by Lieuwen [4]. In the review article of Lieuwen et al. [5], a basic equation is outlined that describes the noise from combustion, which is described as unsteady heat release. The equation for acoustic pressure, p', of Lieuwen et al. [5] is

$$p'(\boldsymbol{x}, t) = \frac{\gamma-1}{4\pi c_\infty^2} \int_{-\infty}^{\infty} \frac{1}{r} \frac{\partial q'}{\partial \tau} d\boldsymbol{y}, \tag{1}$$

where q' is the unsteady heat release per unit volume, r is propagation distance, c_∞ is the ambient speed of sound, t is time, $\boldsymbol{x}$ is the observer position, $\boldsymbol{y}$ is the spatial region of combustion, and τ is source time (retarded time). This equation does not account for other sources of noise that are unique to combustion or turbulence. Lieuwen et al. [5] note that additional sources of waves beyond heat release that are not accounted for by this equation are as follows: steady heat release and coupling of acoustic, vortical, and entropy modes; waves that obliquely impinge at an incident angle on the flame; density gradients and fluctuating pressure gradients that are nonparallel; the unsteady wrinkling of the flame front by acoustic waves; and entropy and vorticity disturbances that interact with a wave where the flame phase speed is supersonic. This research overcomes some of these limitations. Here, we quantify the magnitude of the different acoustic sources relative to other sources. There is a large body of research involving the thermoacoustic source and statistics of corresponding radiating waves. There are many studies on the nature of turbulence and turbulent flows with combustion that involve sound generation (for examples see Bray [6], Roberts and Leventhall [7], Clavin and Siggia [8], Kilham and Kirmani [9], Bailly et al. [10], or Smith and Kilham [11]). Swaminathan et al. [12], Liu [13], and Liu et al. [14] examined heat release, correlations, and sound radiation from turbulent premixed flames. Swirling flame studies of Wasle [15] and Winkler et al. [16] examined the heat release and acoustic efficiency of premixed flames. A number of Japanese studies by Kotake and Takamoto [17], Kotake [18], and Kotake and Takamoto [19] examined combustion noise and its relation to the shape of the nozzle, the size of the burner nozzle, and chemical reactions. Pressure waves and their refraction were examined by Chu [20] and pressure waves generated by a gaseous mixture by Strahle [21].

Overviews of the statistics and radiation of acoustic waves from combustion are presented by Dowling [22], Strahle [23], and Briffa et al. [24]. Klein et al. [25] and Ihme et al. [26] studied noise from nonpremixed flames. A number of source studies of premixed flames from the aeroacoustics community included Jones [27], Strahle and Shivashankara [28], Hirsch et al. [29], Hurle et al. [29], and Rajaram and Lieuwen [30, 31]. Notable models and prediction methods of the noise from combustion included Wasle et al. [32], Liu et al. [33], and most recently Tam [34]. These models used mainly empirical equations with coefficients optimized to fit experimental databases or alternatively used extremely expensive LES without source models.

A number of investigators examined instability in premixed combustion (e.g., Hurle et al. [35]; Price et al. [28], and Strahle [28]). One condition proposed to predict the onset of combustion instability is the Rayleigh criterion. It involves the phased coupling of unsteady heat release with pressure. Lord Rayleigh [36] wrote

"If heat be periodically communicated to, and abstracted from, a mass of air vibrating (for example) in a cylinder bounded by a piston, the effect produced will depend upon the phase of the vibration at which the transfer of heat takes place. If heat be given to the air at the moment of greatest condensation, or be taken from it at the moment of greatest rarefaction, the vibration is encouraged. On the other hand, if heat be given at the moment of greatest rarefaction, or abstracted at the moment of greatest condensation, the vibration is discouraged."

One simplified form of the preceding statement of Rayleigh is

$$\mathscr{R} = \int_\Omega \frac{p\dot{q}}{\bar{\rho} c_p \bar{T}} dV, \tag{2}$$

where $\dot{q}$ is the heat release per unit volume. $\mathscr{R}$ is a dimensional number, whose value when positive implies that there may be combustion instability. However, there are reacting flows where $\mathscr{R} > 0$ and the flow remains stable.

Similar to an acoustic analogy approach, Chu [37] proposed a wave equation with heat source addition as

$$\frac{\partial^2}{\partial t}\left(\frac{p'}{\gamma p_\infty}\right) - c_\infty^2 \nabla^2 \left(\frac{p'}{\gamma p_\infty}\right) = \frac{\partial}{\partial t}\left(\frac{q}{c_p T_\infty}\right), \tag{3}$$

where q is the rate of heat release per unit mass. Chu [37, 20] showed solutions of this equation using one-dimensional examples. A solution in free-space is derived using the free-space Green's function of the wave equation:

$$p'(\gamma p_\infty)^{-1} = \left(4\pi c_\infty^2\right)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r^{-1} \frac{\partial}{\partial t}\left[\frac{q\left(\xi, \eta, \zeta, t - r c_\infty^{-1}\right)}{c_p T_\infty}\right] d\boldsymbol{y}, \tag{4}$$

where r is the distance from the source to observer. The solution for a point source is

$$p'(\gamma p_\infty)^{-1} = \left(4\pi c_\infty^2 r\right)^{-1} \frac{\partial}{\partial t}\left[\frac{Q\left(t - r c_\infty^{-1}\right)}{c_p T_\infty}\right], \tag{5}$$

where $q = \delta Q$.

A number of instability prediction methods use source term analysis. Noiray et al. [38] developed a nonlinear combustion instability analysis. The analysis involves modeling the acoustic propagation properties within a duct, modeling a series of flames at a flame holder plate as monopoles, and finding the dispersion relation. The important point involves the flame amplification factor, $\mathscr{F}$, where an incoming reflected acoustic wave is now dependent on both the frequency and u' (velocity) perturbation. This is the key difference between their traditional linear analysis and the present nonlinear analysis.

NASA SP-194 [39] remains a landmark publication on the topic of combustion instability. Crocco's method uses the time lag of combustion in conjunction with changes in pressures and velocity. Priem's approach examines the actual chemical kinetics of the flow but does not include a lag. Unfortunately, NASA SP-194 [39] does not show development of statistical sources of combustion noise. Yang and Anderson [40] edited a similar compilation in 1995 that is likely inspired by NASA SP-194 [39]. Yang and Anderson's [40] important contribution in Chapter 13 shows a coupling between a wave equation acoustic model and the acoustic modes within the flow. They decompose the Helmholtz equation into a number of modes with corresponding right-hand side function F_n. The function F_n is divided into a number of source components. It accounts for the linear and nonlinear effects due to the coupling of the acoustics with the mean flow. The remaining portion of their analysis shows the effects of acoustic mode coupling. A similar frequency domain modal decomposition technique using the full equations of motion is proposed in this paper (instead of the Helmholtz equation).

Mitchell [41] (within Yang and Anderson [40]) discusses analytical models for combustion instability. Likewise, the system is solved with the Green's function of the Helmholtz equation. Kim [42] in Yang and Anderson [40] discuss the effect of turbulence on the thermoacoustic instability. This is unlike the work of Mitchel who neglected such effects with the formation of the Helmholtz equation. Kim [42] essentially used a stochastic method applied to the governing equations and mass fraction equations. Like the previous approaches ([40, 41]), a wave equation solution was found. The interaction of turbulence and thermoacoustic instability can only be analyzed at the source in the model.

Jacob and Batterson [43] created a theory based on the premise of instability manifesting as periodic behavior such as acoustic resonance or hydrodynamic vortex shedding. Their approach is based on decomposing the flow-field into a steady and unsteady modal fluctuation. Jacob and Batterson [43] argue that there is a modified Rayleigh criterion (see their Equation (26)). A puzzling aspect of the work of Jacob and Batterson [43], and works similar that have come before, resides in the fact that all acoustic energy radiates into modes of the geometric acoustic field. This is not physically possible, as acoustic energy from turbulence (and reacting turbulent flows) radiates at all wavenumbers and has strong directivity. The decomposition used in the present work does not make such an assumption.

2. Mathematical Theory

Our approach is based on the work of Miller [44]. We extend this approach to include combustion sources within a premixed turbulent field. The method depends on a single set of partial differential equations that are of order $N + 5$, where N is the number of species. Our base equations are the following: the continuity equation for conservation of mass; the momentum equation with additional term $\rho\sum_{k=1}^{N} Y_k f_{k,i}$ for conservation of momentum, where Y_k is the mass fraction of species k and $f_{k,i}$ is the volume force acting on species k; and the sensible energy equation e_s, with right-hand side involving $\rho\sum_{k=1}^{N} h_{s,k} Y_k V_{k,i}$ and $\rho\sum_{k=1}^{N} Y_k f_{k,i} V_{k,i}$, where h_s is the sensible enthalpy and V is the diffusion velocity of species k. Through an ideal gas law $p_k = \rho_k R T W_k^{-1}$, we will write our energy equation for e_s in terms of pressure. Here, W_k is the molecular weight of species k.

We seek a closed-form solution for the outgoing fluctuating waves in the form of an integral equation. Let $\boldsymbol{q}$ be the vector of the field-variables consisting of $\rho, \boldsymbol{u}, p$, and Y_k, for $k = 1$ to N. We now decompose the equations with $\boldsymbol{q} = \bar{\boldsymbol{q}} + \hat{\boldsymbol{q}} + \boldsymbol{q}'$, where the overline represents the base flow or time-average, the hat represents the fluctuating variable, and prime denotes the fluctuations about the base flow (overline) and turbulent combustion (hat). The fluctuating quantities involving outgoing waves are moved to the left-hand side of the equations, and the remaining quantities are moved to the right-hand side. We solve these equations for $\boldsymbol{q}'$ using the method of the vector Green's function. This results in a closed-form equation involving the vector Green's function of the linearized Navier-Stokes equations (NSE) including chemical reactions for each species k convolved with the NSE with chemical reactions acting on $\bar{\boldsymbol{q}}$ and $\boldsymbol{q}^{\sim}$.

Using these equations, we find a stochastic closed-form prediction model for the acoustic fluctuation p'. We form a spectral density equation for $\boldsymbol{q}$ by performing a two-point space-time cross-correlation of the source terms. A subsequent inverse Fourier transform results in the spectral density, $\boldsymbol{S}(\boldsymbol{x}, \omega)$ of $\boldsymbol{q}$. This statistical source formulation can be used to assess LES or DNS databases to detect the onset, magnitude, and exact source statistics of thermoacoustic instabilities.

3. Governing Equations

We use the Navier-Stokes equations, the diffusion-velocity equations, an energy equation written in terms of $\partial p/\partial t$, and additional closure equations such as gas laws to model the flow-field. The continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \tag{6}$$

where u_j is the velocity, x is the spatial coordinate, t is the time, and ρ is the density. Alternatively, we can use the mass conservation for species k:

$$\frac{\partial \rho Y_k}{\partial t} + \frac{\partial}{\partial x_j}\left[\rho\left(u_j + V_{k,j}\right)Y_k\right] = \dot{\omega}_k, \tag{7}$$

where $V_{k,j}$ is the j^{th} component of the diffusion velocity V_k of species k, $\dot{\omega}_k$ is its associated reaction rate, and Y_k are the mass fractions of species k as $Y_k = m_k m^{-1}$. Here, m_k is the mass of species k and m is the total mass in the fluid parcel. The momentum equation is

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho \sum_{k=1}^{N} Y_k f_{k,i}, \tag{8}$$

where $f_{k,i}$ is the body force per unit volume of species k, δ_{ij} is the Kronecker delta function, and p is the pressure. The shear stress, τ_{ij}, is

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}, \tag{9}$$

where μ is the viscosity. The energy equation in terms of pressure, p, is

$$\begin{aligned}
&\frac{\partial p}{\partial t} + \frac{\gamma - 1}{2}\frac{\partial \rho u_k u_k}{\partial t} + \frac{\gamma - 1}{2}\frac{\partial \rho u_j u_k u_k}{\partial x_j} \\
&\quad = -\gamma\frac{\partial u_j p}{\partial x_j} + (\gamma - 1)\dot{\omega}_T' + (\gamma - 1)\frac{\partial}{\partial x_j}\left[\lambda\frac{\partial T}{\partial x_j}\right] \\
&\qquad + (\gamma - 1)\frac{\partial u_i \tau_{ij}}{\partial x_j} + (\gamma - 1)\dot{Q} + (\gamma - 1)\rho\sum_{k=1}^{N} Y_k f_{k,i} V_{k,i},
\end{aligned} \tag{10}$$

where $\dot{Q}$ is a heat source (e.g., plasma, spark, and laser) and γ is the ratio of specific heats. 4The thermal conductivity $\lambda = c_p \mu \,\mathrm{Pr}^{-1}$, where Pr is the Prandtl number. Also, $\dot{\omega}_T' = -\sum_{k=1}^{N} h_k \dot{\omega}_k$, where h_k is the enthalpy of species k. We introduce the equation for h_k

$$h_k = h_{sk} + \Delta h_{f,k}^o = \int_{T_o}^{T} c_{p,k}\,dT + \Delta h_{f,k}^o, \tag{11}$$

which is the summation of sensible enthalpy and chemical enthalpy. The diffusion velocity of species k is now

$$V_k = -D_k\frac{\nabla X_k}{X_k} + V_c = -D_k\frac{\nabla X_k}{X_k} + \frac{1}{W}\sum_{k=1}^{N} D_k W_k \nabla X_k, \tag{12}$$

where the correction velocity is $V_c = \sum_{k=1}^{N} D_k (W_k/W)(\partial X_k/\partial x_i)$. We use the Hirschfelder and Curtiss approximation, which approximates the exact system of Williams. The diffusion-velocity equation is

$$\frac{\partial \rho Y_k}{\partial t} + \frac{\partial \rho u_j Y_k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\rho D_k \frac{W_k}{W}\frac{\partial X_k}{\partial x_j}\right) + \dot{\omega}_k, \tag{13}$$

where $\dot{\omega}_k$ is the reaction rate of species k. D_k is an equivalent diffusion coefficient of species k into the surrounding mixture

$$D_k = \frac{1 - Y_k}{\sum_{j\neq k} X_j/\mathscr{D}_{jk}} \tag{14}$$

and $X_k = Y_k W W_k^{-1}$ is the mole fraction of species k. W is the molecular weight of the mixture and W_k is the molecular weight of the species k, related by $W^{-1} = \sum_{k=1}^{N} Y_k W_k^{-1}$. $\mathscr{D}_{k,y}$ is the diffusion coefficient of species k into species y.

4. Decomposition

We decompose our equations of motion into a summation of components of field-variables. We define a vector $\boldsymbol{q} = \boldsymbol{q}(\rho, \boldsymbol{u}, p, Y_k)$ as the vector of field-variables. This choice of decomposition is critical. We elect to make the simplest decomposition for this purpose, which involves a base flow, turbulent fluctuation, and acoustic perturbation. The decomposition is

$$\boldsymbol{q} = \bar{\boldsymbol{q}} + \hat{\boldsymbol{q}} + \boldsymbol{q}', \tag{15}$$

where the overline represents the base flow, the hat operator represents the fluctuating turbulent quantity, and the perturbation represents the fluctuating acoustic quantity. More complicated decompositions have previously been made that further divide the turbulence based on statistics and associated radiated waves.

5. Source Terms of Waves

These decomposed field-variables are substituted into our governing equations. The resulting equations are rearranged such that the radiating terms are on the left-hand side. The time-averaged base flow and fluctuations are on the right-hand side. The right-hand side of the equations results in the exact NSE operator on the summation of the base flow and turbulent fluctuations. The left-hand side terms are viewed as propagators, while the right-hand side terms are source terms. The continuity source term is

$$\Theta_0 = -\frac{\partial \rho}{\partial t} - \frac{\partial \underline{\rho} \underline{u}_j}{\partial x_j} = 0. \tag{16}$$

The momentum source term (for $i = 1$ to 3) is

$$\Theta_i = -\frac{\partial \rho \underline{u}_i}{\partial t} - \frac{\partial \rho \underline{u}_i \underline{u}_j}{\partial x_j} - \frac{\partial \underline{p}}{\partial x_j}\delta_{ij} + \frac{\partial \tau_{ij}}{\partial x_j} + \underline{\rho}\sum_{k=1}^{N} \underline{Y}_k f_{k,i}. \tag{17}$$

The energy source term is

$$\begin{aligned}\Theta_4 = &-\frac{\partial p}{\partial t} - \frac{\gamma-1}{2}\frac{\partial \rho \underline{u}_k \underline{u}_k}{\partial t} - \gamma \frac{\partial \underline{u}_j \underline{p}}{\partial x_j} - \frac{\gamma-1}{2}\frac{\partial \rho \underline{u}_j \underline{u}_k \underline{u}_k}{\partial x_j} \\ &+ (\gamma-1)\underline{\dot{\omega}}_T' + (\gamma-1)\frac{\partial}{\partial x_j}\left[\lambda \frac{\partial \underline{T}}{\partial x_j}\right] + (\gamma-1)\frac{\partial \underline{u}_i \tau_{ij}}{\partial x_j} \\ &+ (\gamma-1)\,\dot{Q} + (\gamma-1)\underline{\rho}\sum_{k=1}^{N} \underline{Y}_k f_{k,i} \underline{V}_{k,i}.\end{aligned} \tag{18}$$

The diffusion-velocity source term is

$$\Theta_5 = -\frac{\partial \rho \underline{Y}_k}{\partial t} - \frac{\partial \rho \underline{u}_j \underline{Y}_k}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\varrho \underline{D}_k \frac{W_k}{W}\frac{\partial \underline{X}_k}{\partial x_j}\right) + \underline{\dot{\omega}}_k. \tag{19}$$

Here, Θ_0 represents the source terms from the continuity equation, Θ_i (where $i = 1, 2, 3$) represents the three components of the source terms from the momentum equations, and Θ_4 represents the source terms from the energy equation. The under-bar operator denotes the sum of the base quantity and fluctuating turbulent quantities, i.e., $\underline{q} = \bar{q} + \hat{q}$.

6. General Solution for Radiated Waves in Temporal and Spectral Domains

It is difficult to find the solution if the system of equations contains Green's function involving the velocity diffusion equation. For only the purpose of evaluating the vector Green's function analytically, we make certain assumptions. We assume that the propagation is decoupled from the source so that the method of vector Green's function can be used. The species mass fraction is relatively constant (if premixed) in the region of propagation outside the source (if localized). The value of $\dot{\omega}_k$ is zero outside the region of the turbulent reacting flow, and D_k is also constant in this region. The molecular weights are not changing on a species basis W_k. The molar fraction of species, k, X_k, is also constant in the acoustic medium spatially. We can then approximate $\partial X_k/\partial x_j$ as zero in the propagation region.

Thus, the mass fraction equation can be decoupled with the other equations for the purposes of only finding the vector Green's function. This is equivalent to other approaches that define acoustic propagation via a Helmholtz equation. Using the Wiener-Khinchin theorem [45], we can define the spectral density as the inverse Fourier transform of the autocorrelation function. Using the definition of autocorrelation, we write the spectral density of pressure as

$$\begin{aligned}S_4(x,\omega) = &\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{m=0}^{4}\sum_{n=0}^{4}\tilde{p}_G^{*,m}(x\,;y,\omega)\tilde{p}_G^{n}(x\,;y+\eta,\omega) \\ &\cdot R_{m,n}(y,\eta,\tau)\exp\left[i\omega\tau\right]d\tau d\eta dy.\end{aligned} \tag{20}$$

The two-point space-time cross-correlation of the source terms is denoted by $R_{m,n}(y, \boldsymbol{\eta}, \tau)$ and written as

$$\begin{aligned}R_{m,n}^{\perp}(y,\eta,\tau) &= \langle \Theta_m(y,\tau)\Theta_n(y+\eta,\tau+\Delta\tau)\rangle \\ &= \int_{-\infty}^{\infty}\Theta_m(y,\tau)\Theta_n(y+\eta,\tau+\Delta\tau)d\Delta\tau,\end{aligned} \tag{21}$$

where $\boldsymbol{\eta} = (\xi, \eta, \zeta)$ is a source separation vector pointing from one source location to the other and $\Delta\tau$ is the time delay between the two sources.

7. Vector Green's Function

The propagators on the left-hand side are linearized. Miller [44] derived the solution to the linearized continuity, momentum, and energy equations using the vector Green's function, which satisfies the linearized continuity equation

$$\frac{\partial \rho_G^n}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho}u_{j,G}^n + \rho_G^n \bar{u}_j\right) = \delta(x-y)\delta(t-\tau)\delta_{0n}, \tag{22}$$

the linearized momentum equation

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\bar{\rho}u_{i,G}^n + \rho_G^n \bar{u}_i\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}u_i u_{j,G}^n + \bar{\rho}u_{i,G}^n \bar{u}_j + \rho_G^n \bar{u}_i \bar{u}_j\right) \\ &\quad + \frac{\partial p_G^n}{\partial x_j}\delta_{ij} - \frac{\partial \bar{\tau}_{ij}}{\partial x_j} - \rho\sum_{k=1}^{N}\left(\bar{Y}_k + Y_{k,G}^n\right)f_{k,i} = \delta(x-y)\delta(t-\tau)\delta_{\mathrm{in}},\end{aligned} \tag{23}$$

the linearized energy equation

$$\begin{aligned}&\frac{\partial p_G^n}{\partial t} + \frac{\gamma-1}{2}\frac{\partial}{\partial t}\left(\rho_G^n \bar{u}_k \bar{u}_k + 2\bar{\rho}u_{k,G}^n \bar{u}_k\right) \\ &\quad + \frac{\gamma-1}{2}\frac{\partial}{\partial x_j}\left(\rho_G^n \bar{u}_j \bar{u}_k \bar{u}_k + \bar{\rho}u_{j,G}^n \bar{u}_k \bar{u}_k + 2\bar{\rho}u_j u_{k,G}^n \bar{u}_k\right) \\ &\quad + \gamma\frac{\partial}{\partial x_j}\left(u_{j,G}^n \bar{p} + \bar{u}_j p_G^n\right) - (\gamma-1)\bar{\omega}_T' - (\gamma-1)\frac{\partial}{\partial x_j}\left[\bar{\lambda}\frac{\partial \bar{T}}{\partial x_j}\right] \\ &\quad - (\gamma-1)\frac{\partial\left(\bar{u}_i + u_{i,G}^n\right)\bar{\tau}_{ij}}{\partial x_j} - (\gamma-1)\bar{Q} - (\gamma-1)\left(\bar{\rho} + \rho_G^n\right) \\ &\quad \cdot \sum_{k=1}^{N}\left(\bar{Y}_k + Y_{k,G}^n\right)\bar{f}_{k,i}\bar{V}_{k,i} = \delta(x-y)\delta(t-\tau)\delta_{4n},\end{aligned} \tag{24}$$

and the linearized mass fraction equation

$$\frac{\partial\left(\bar{Y}_k\rho_G^n + \bar{\rho}Y_{k,G}^n\right)}{\partial t} + \frac{\partial\bar{\rho}\bar{u}_j\bar{Y}_k + \bar{u}_j\bar{Y}_k\rho_G^n + \bar{\rho}\bar{Y}_k u_{j,G}^n + \bar{\rho}\bar{u}_j Y_{k,G}^n}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\left(\bar{\rho} + \rho_G^n\right)\bar{D}_k\frac{\bar{W}_k}{\bar{W}}\frac{\partial\bar{X}_k}{\partial x_j}\right) - \dot{\bar{\omega}}_k = \delta(x-y)\delta(t-\tau)\delta_{5n}. \tag{25}$$

The vector Green's function, $q_{k,g}^{\perp n} = [\rho_g^{\perp n}, \boldsymbol{u}_{i,g}^{\perp n}, p_g^{\perp n}, Y_{k,g}^{\perp n}]^T$, is written in terms of the vector field-variable. The subscript g denotes Green's function, $\boldsymbol{x}$ and $\boldsymbol{y}$ represent the observer location and source location, respectively, $\delta()$ is the Dirac delta function, and τ is the source emission time. The refraction effects for a shear layer in the sideline direction are minimal (see for example [46, 47], or [48]). To obtain the vector Green's function for pressure, we perform the partial time derivative on the energy Green's function and perform the gradient on the momentum Green's function. The result of these operations is an inhomogeneous wave equation. The Fourier transform is performed on the inhomogeneous wave equation, and the solution to the vector Green's function for pressure can be found:

$$\begin{aligned}\tilde{p}_G^n = {} & \left(\frac{i\omega}{4\pi c_\infty R} - \frac{1}{4\pi R^2}\right)\frac{(x_i - y_i)}{R}\exp\left[\frac{i\omega R}{c_\infty}\right]\delta_{\text{in}} \\ & - \frac{i\omega}{4\pi c_\infty^2 R}\exp\left[\frac{i\omega R}{c_\infty}\right]\delta_{4n},\end{aligned} \tag{26}$$

where $R = |\boldsymbol{x} - \boldsymbol{y}|$.

8. Two-Point Expansions

We show the expansions of $\langle\Theta_i^{(1)}, \Theta_l^{(2)}\rangle$ for $i = 0$ to 5 and $l = 0$ to 5. The superscript (1) or (2) denote the first and second positions of the two-point cross-correlation, respectively. A compact notation is used in each subsection of this appendix. Each term of the expansion involves superscript (1) or (2), which denotes the terms associated with their respective point. An R is appended to each term that represents the normalized form of the two-point cross-correlation. The hat operator on R denotes that the correlation acts on the fluctuating scales. This is redundant in this expansion but is important in more complicated derivations involving multiple scales. The magnitude of the correlation is the term itself because the maximum norm of $\widehat{R}$ is unity. For example, the expansion of the first term of continuity and the second term of the momentum equation maps as

$$\left\langle\frac{\partial\rho}{\partial\tau}, \frac{\partial\rho u_i u_j}{\partial x_j}\right\rangle \longrightarrow \frac{\partial\rho^{(1)}}{\partial\tau}\frac{\partial\rho^{(2)}u_i^{(2)}u_j^{(2)}}{\partial x_j}\widehat{R}. \tag{27}$$

8.1. Sources Unique to Combustion Noise. Here, we identify sources that are unique to combustion in the context of the proposed model.

We then identify sources that are most likely dominant based on aeroacoustic theory. Terms unique to combustion are $\Theta_{i,5}$ for $i = 1$ to 3, $\Theta_{4,5}$,$\Theta_{4,8}$, $\Theta_{4,9}$, and $\Theta_{5,j}$ for $j = 1$ to 4. Structures of turbulence that are reacting and create noise have coherence in space and time. Combustion terms involving two-point cross-correlations with themselves are $\Theta_{1,5}$, $\Theta_{2,5}$, $\Theta_{3,5}$, $\Theta_{4,5}$, $\Theta_{4,8}$, $\Theta_{4,9}$, $\Theta_{5,1}$, $\Theta_{5,2}$, $\Theta_{5,3}$, and $\Theta_{5,4}$. These source terms are only dependent on the combustion terms within the equations of motion. We write them as

$$\mathcal{T}_{\text{comb}} = [\Theta_{1,5}, \Theta_{2,5}, \Theta_{3,5}, \Theta_{4,5}, \Theta_{4,8}, \Theta_{4,9}, \Theta_{5,1}, \Theta_{5,2}, \Theta_{5,3}, \Theta_{5,4}]. \tag{28}$$

The radiating waves can be predicted in either the time-domain or within a power spectral domain. It is customary to quantify statistical sources in the spectral domain. All sources of noise within a turbulent flow have some amount of spatial or temporal coherence. We write the two-point space-time cross-correlations (of varying order depending on the number of terms) of the purely combustion source terms as

$$\left\langle\mathcal{T}_{\text{comb}}^{(1)}, \mathcal{T}_{\text{comb}}^{(2)}\right\rangle, \tag{29}$$

where superscripts (1) and (2) represent the points of the two-point cross-correlation. These correlations can be expanded about the base flow and the perturbation. Two-point cross-correlations of combustion terms with traditional aerodynamic terms are

$$\left\langle\mathcal{T}_{\text{Comb}}^{(1)}, \mathcal{T}_{\text{Aero}}^{(2)}\right\rangle \text{ and } \left\langle\mathcal{T}_{\text{Comb}}^{(2)}, \mathcal{T}_{\text{Aero}}^{(1)}\right\rangle. \tag{30}$$

We now define $\mathcal{T}_{\text{comb}}^{(1)}$:

$$\begin{aligned}\mathcal{T}_{\text{comb}}^{(1)} = {} & \left[\underline{\rho}^{(1)}\sum_{n=1}^{N}\underline{Y}_k^{(1)}f_{k,i}^{(1)}, (\gamma-1)\underline{\dot{\omega}}_T'(1), (\gamma-1)\underline{\rho}^{(1)}\right. \\ & \cdot\sum_{k=1}^{N}\underline{Y}_k^{(1)}f_{k,i}^{(1)}\underline{V}_{k,i}^{(1)} - \frac{\partial\underline{\rho}^{(1)}\underline{Y}_k^{(1)}}{\partial t}, -\frac{\partial\rho^{(1)}\underline{u}_j^{(1)}\underline{Y}_k^{(1)}}{\partial x_j}, \frac{\partial}{\partial x_j} \\ & \left.\cdot\left(\frac{\rho^{(1)}}{\underline{D}_k^{(1)}}\frac{W_k^{(1)}}{W^{(1)}}\frac{\partial\underline{X}_k}{\partial x_j^{(1)}}\right), \underline{\dot{\omega}}_n^{(1)}\right]^T,\end{aligned} \tag{31}$$

where a similar form exists for $\mathcal{T}^{(2)}_{\text{comb}}$. We find the resultant combustion-acoustic source terms as

$$\begin{gathered}
(\gamma-1)^2\left\langle \dot{\underline{\omega}}'_T,(1), \dot{\underline{\omega}}'_T,(2)\right\rangle, -(\gamma-1)\left\langle \underline{\omega}'_T,(1), \frac{\partial \rho^{(2)}\underline{Y}_n^{(2)}}{\partial t}\right\rangle, \\
-(\gamma-1)\left\langle \underline{\omega}'_T,(1), \frac{\partial \rho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle, (\gamma-1)\left\langle \underline{\omega}'_T,(1), \underline{\omega}_n^{(2)}\right\rangle, \\
\left\langle \frac{\partial \rho^{(1)}\underline{Y}_k^{(1)}}{\partial t}, \frac{\partial \rho^{(2)}\underline{Y}_n^{(2)}}{\partial t}\right\rangle, \left\langle \frac{\partial \varrho^{(1)}\underline{Y}_k^{(1)}}{\partial t}, \frac{\partial \varrho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle, \\
-\left\langle \frac{\partial \rho^{(1)}\underline{Y}_k^{(1)}}{\partial t}, \underline{\omega}_n^{(2)}\right\rangle, \left\langle \frac{\partial \rho^{(1)}\underline{u}_j^{(1)}\underline{Y}_k^{(1)}}{\partial x_j}, \frac{\partial \rho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle, \\
-\left\langle \frac{\partial \underline{\rho}^{(1)}\underline{u}_j^{(1)}\underline{Y}_k^{(1)}}{\partial x_j}, \dot{\underline{\omega}}_n^{(2)}\right\rangle, \text{and} \left\langle \dot{\underline{\omega}}_k^{(1)}, \dot{\underline{\omega}}_n^{(2)}\right\rangle.
\end{gathered} \tag{32}$$

Two-point source correlation terms involving traditional aeroacoustic sources and combustion sources, $\langle \mathcal{T}^{(1)}_{\text{Aero}}, \mathcal{T}^{(2)}_{\text{comb}}\rangle$, are

$$\begin{gathered}
-(\gamma-1)\left\langle \frac{\partial \rho^{(1)}\underline{u}_i^{(1)}\underline{u}_j^{(1)}}{\partial x_j}, \dot{\underline{\omega}}'_T,(2)\right\rangle, \left\langle \frac{\partial \varrho^{(1)}\underline{u}_i^{(1)}\underline{u}_j^{(1)}}{\partial x_j}, \frac{\partial \rho^{(2)}\underline{Y}_n^{(2)}}{\partial t}\right\rangle, \\
\left\langle \frac{\partial \rho^{(1)}\underline{u}_i^{(1)}\underline{u}_j^{(1)}}{\partial x_j}, \frac{\partial \rho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle, -\left\langle \frac{\partial \underline{\rho}^{(1)}\underline{u}_i^{(1)}\underline{u}_j^{(1)}}{\partial x_j}, \underline{\underline{\omega}}_n^{(2)}\right\rangle, \\
-(\gamma-1)\left\langle \frac{\partial p^{(1)}}{\partial x_j}\delta_{ij}, \underline{\omega}'_T,(2)\right\rangle, -\left\langle \frac{\partial p^{(1)}}{\partial x_j}\delta_{ij}, \dot{\underline{\omega}}_n^{(2)}\right\rangle, \\
(\gamma-1)\left\langle \frac{\partial p^{(1)}}{\partial t}, \dot{\underline{\omega}}'_T,(2)\right\rangle, \left\langle \frac{\partial p^{(1)}}{\partial x_j}\delta_{ij}, \frac{\partial \varrho^{(2)}\underline{Y}_n^{(2)}}{\partial t}\right\rangle \\
\cdot\left\langle \frac{\partial p^{(1)}}{\partial x_j}\delta_{ij}, \frac{\partial \varrho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle, \left\langle \frac{\partial p^{(1)}}{\partial t}, \frac{\partial \rho^{(2)}\underline{Y}_n^{(2)}}{\partial t}\right\rangle, \\
\left\langle \frac{\partial p}{\partial t}, \frac{\partial \rho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle, -\left\langle \frac{\partial p^{(1)}}{\partial t}, \dot{\underline{\omega}}_n^{(2)}\right\rangle, \\
-\frac{(\gamma-1)^2}{2}\left\langle \frac{\partial \rho^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial t}, \dot{\underline{\omega}}'_T(2)\right\rangle, \\
\frac{\gamma-1}{2}\left\langle \frac{\partial \rho^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial t}, \frac{\partial \rho^{(2)}\underline{Y}_n^{(2)}}{\partial t}\right\rangle, \\
\frac{\gamma-1}{2}\left\langle \frac{\partial \rho^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial t}, \frac{\partial \rho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle, \\
-\frac{\gamma-1}{2}\left\langle \frac{\partial \rho^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial t}, \dot{\underline{\omega}}_n^{(2)}\right\rangle \\
-(\gamma(\gamma-1))\left\langle \frac{\partial \underline{u}_j^{(1)}p^{(1)}}{\partial x_j}, \underline{\omega}'_T,(2)\right\rangle, \gamma\left\langle \frac{\partial \underline{u}_5^{(1)}p^{(1)}}{\partial x_j}, \frac{\partial \rho^{(2)}\underline{Y}_n^{(2)}}{\partial t}\right\rangle, \\
\gamma\left\langle \frac{\partial \underline{u}_j^{(1)}p^{(1)}}{\partial x_j}, \frac{\partial \rho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle - \gamma\left\langle \frac{\partial \underline{u}_j^{(1)}\underline{p}^{(1)}}{\partial x_j}, \dot{\underline{\omega}}_n^{(2)}\right\rangle, \\
-\frac{(\gamma-1)^2}{2}\left\langle \frac{\partial \varrho^{(1)}\underline{u}_j^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial x_j}, \dot{\underline{\omega}}'_T,(2)\right\rangle, \\
\frac{\gamma-1}{2}\left\langle \frac{\partial \rho^{(1)}\underline{u}_j^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial x_j}, \frac{\partial \rho^{(2)}\underline{Y}_n^{(2)}}{\partial t}\right\rangle \\
\left\langle \frac{\gamma-1}{2}\frac{\partial \rho^{(1)}\underline{u}_j^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial x_j}, \frac{\partial \rho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle, \\
-\left\langle \frac{\gamma-1}{2}\frac{\partial \rho^{(1)}\underline{u}_j^{(1)}\underline{u}_k^{(1)}\underline{u}_k^{(1)}}{\partial x_j}, \dot{\underline{\omega}}_n^{(2)}\right\rangle, (\gamma-1)^2\left\langle \underline{Q}^{(1)}, \dot{\underline{\omega}}'_T(2)\right\rangle, \\
-(\gamma-1)\left\langle \underline{Q}^{(1)}, \frac{\partial \rho^{(2)}\underline{Y}_n^{(2)}}{\partial t}\right\rangle - (\gamma-1)\left\langle \underline{Q}^{(1)}, \frac{\partial \varrho^{(2)}\underline{u}_m^{(2)}\underline{Y}_n^{(2)}}{\partial x_m}\right\rangle, \\
(\gamma-1)\left\langle \dot{\underline{Q}}^{(1)}, \dot{\underline{\omega}}_n^{(2)}\right\rangle.
\end{gathered} \tag{33}$$

Note that we moved γ outside of the operation for simplicity of writing the vectors, but they can easily be moved inside <,> to remain local.

8.2. Source Analysis—Modified Rayleigh Criterion. We now compare our derived sources relative to previous theories involving combustion-acoustic sources. We seek to find alternative forms of the classic Rayleigh criterion, $\mathcal{R}$, in the form similar to that proposed by Chu [20] introduced in Equation (2). Equation (2) involves the fluctuations of p with the time derivative of heat release. There are no terms that involve p or $p\dot{q}$ within $\mathcal{T}^{(1)}_{\text{comb}}$. However, terms do appear in $\langle \mathcal{T}^{(1)}_{\text{Aero}}, \mathcal{T}^{(2)}_{\text{comb}}\rangle$ involving p and the unsteady heat release. These are $-(\gamma-1)\langle(\partial \underline{p}^{(1)}/\partial x_j)\delta_{ij}, \dot{\underline{\omega}}'_T,(2)\rangle$ and $-(\gamma-1)\langle(\partial \underline{p}^{(1)}/\partial t), \dot{\underline{\omega}}'_T,(2)\rangle$. Let us examine these terms and drop the superscript position index for simplicity. The first can be expanded, and we eliminate the base flow term of pressure with respect to time and expand the second term, which becomes

$$-(\gamma-1)\left\langle \sum_{i=1}^{3}\frac{\partial \hat{p}}{\partial x_j}\delta_{ij}, \widehat{\dot{\omega}}'_T\right\rangle \text{and} - (\gamma-1)\left\langle \frac{\partial \hat{p}}{\partial t}, \widehat{\omega}'_T\right\rangle. \tag{34}$$

The first term involves the divergence of $\hat{p}$ with the time derivative of the unsteady heat release, which is similar to the integrand of $\mathcal{R}$. The second term involves the time derivative of the fluctuation pressure $\hat{p}$ and the time derivative of the unsteady heat release, which is also similar to $\mathcal{R}$. For these reasons, we propose the following modified $\mathcal{R}$ criterion:

$$\mathcal{R}_\alpha = \int_\Omega \left\langle \nabla \hat{p}, \widehat{\dot{\omega}}'_T\right\rangle dV, \tag{35}$$

and

$$\mathscr{R}_\beta = \int_\Omega \left\langle \frac{\partial \hat{p}}{\partial t}, \hat{\omega^{\cdot}}{}'_T \right\rangle dV. \tag{36}$$

Equations (35) and (36) differ in their evaluation relative to Equation (2). $\mathscr{R}_\alpha$ and $\mathscr{R}_\beta$ involve the volumetric integration of a two-point cross-correlation. For each point within the reacting portion of the flow-field, every other point needs to be accounted for in the evaluation of the integrands of $\mathscr{R}_\alpha$ and $\mathscr{R}_\beta$. The major advantage of this approach is that with reacting flow, spatially coherent structures that involve highly coherent aerodynamic pressures and unsteady heat release due to reactants, k, will be powerful radiators of acoustic energy. Highly negatively correlated p and unsteady heat release spatially will be stable (large negative values of $\mathscr{R}$). The evaluation of $\mathscr{R}_\alpha$ and $\mathscr{R}_\beta$ results in a scalar value. Much like a critical Reynolds number that leads to transition, there are critical values of $\mathscr{R}_\alpha$ and $\mathscr{R}_\beta$ for the instability of particular reacting flows. Furthermore, the original approaches proposed by Chu [20] and Rayleigh are only one-point theories, while this approach takes into account the whole spatial domain of the reacting flow-field.

If a reacting flow reflects acoustic waves and they are of sufficient strength, then the terms within Equations (35) and (36) must be modified to include the component of the reflected wave. This likely occurs in an unstable reacting flow undergoing thermoacoustic instability. For example, Equation (35) will have $\hat{p}$ approximated as $\hat{p} + p'$, but such an approach breaks a number of assumptions made for this method and will require a nonlinear approach. One can use Equation (20) to predict the acoustic spectra from these source terms; however, this will not yield any insight into the stability of a reacting flow, but only the radiated noise.

8.3. Source Analysis—Combustion Source Model. In the newly proposed model, the pressure due to the unsteady heat release can be found if the terms within $\mathbf{\Theta}_i$ of Equation (18) are set to zero except the unsteady heat release. Using the appropriate vector Green's function in Equation (26), we find

$$q_k'(x, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{j=0}^{5} G_{j,k}(x, t; y, \tau)\Theta_j(y, \tau) d\tau dy. \tag{37}$$

This equation is a general expression for the radiated waves of q' from a reacting turbulent flow. Substituting in the vector Green's function and the source model and seeking $k = 4$ for acoustic pressure, eliminating terms within the source model that do not involve combustion, performing the integration with respect to ω and τ, we find

$$p'(x, t) \approx \frac{\gamma - 1}{4\pi c_\infty^2} \int_{-\infty}^{\infty} |x - y|^{-1} \left(\underline{\dot{\omega}}_T' + \underline{Q} + \underline{\rho} \sum_{k=1}^{N} \underline{Y}_k f_{k,i} \underline{V}_{k,i} \right) dy. \tag{38}$$

We note that there are only three unique combustion sources in the context of the classical acoustic analogy. If we eliminate the body force term, $f_{k,i}$, and the term involving $\underline{Q}^{\cdot}$ (external point heat source), then we find

$$p'(x, t) \approx \frac{\gamma - 1}{4\pi c_\infty^2} \int_{-\infty}^{\infty} |x - y|^{-1} \left(\bar{\omega^{\cdot}}{}'_T + \hat{\omega^{\cdot}}{}'_T \right) dy. \tag{39}$$

If the base variable is eliminated, then we have recovered the theory of Lieuwen et al. [5] shown in Equation (1).

9. Applications

Two popular methods for thermoacoustic combustion instability prediction are based on RANS and LES analyses (for example, see [49–53]). The methods rely on a CFD solver to find the statistics or unsteady motions of the combustion chamber or entire engine interior. Simplified models use a transfer function (more formally a simplified Green's function governed by equations (22) through (25)) to predict growth or decay of the reacting field via a feedback loop. These same equations can be reduced to the simple wave or convective wave equation used for the transfer function within engines as described by Poinsot et al. [54]. An approach that does not make the assumptions of spherical spreading of thermoacoustic waves can be created (with an appropriate mean flow-field) based on equations (22)–(25). In addition, if an LES of a combustion core is available, then the full source terms shown in equations (16)–(19) should be used. This will overcome the assumption that the thermoacoustic source is driven only by the unsteady heat release in traditional models described by Poinsot et al. [54].

The newly presented theory can be directly applied to the prediction of the noise from combustion within contemporary engines. For example, the core noise [55] from high-bypass turbofan engines has continued to increase beyond levels of other noise sources even during take-off [55]. Currently, prediction methods for core noise at the system level are entirely empirical [56] or based on the work of Chu [37]. Use of Equation (37) with all source terms combined with a modern LES of the core itself and a tailored vector Green's function of the engine geometry and mean flow will produce a prediction without all the assumptions inherent in the NASA Aircraft Noise Prediction Program (ANOPP/2) [57]. Also, a simplified form that will use little modification relative to standard acoustic prediction models appears in Equation (39). Here, the same unsteady heat release model for traditional combustion-acoustic codes can be used with the standard free-space or tailored Green's function. Finally, if a traditional acoustic prediction model is desired, then Equation (38) can be evaluated, which includes two additional terms that were neglected recklessly in classical theory. There are undoubtedly other applications for aeroacoustics, such as in traditional internal combustion engines or for academic studies of flame acoustics.

10. Advantages and Disadvantages of the Present Theory

Here, we state the advantages and disadvantages of the presently derived theory. Advantages are as follows:

(i) All source terms, Θ_i, are derived analytically from the reacting Navier-Stokes equations. They are not empirically ascertained as source terms like in previous approaches (see Lieuwen [4] or Liu [14]). One cannot be sure that all noise or thermoacoustic combustion sources are present when using empirically based approaches. Therefore, the present approach makes no such assumptions

(ii) The full effects of propagation outside the region of reaction are fully accounted for through the vector Green's function that is governed by the linearized Navier-Stokes equations (see equations (22)–(25)). We have presented the quiescent solution of these equations in Equation (26). If certain boundary conditions are used (such as solid or porous walls of an engine), then generally no analytical solution exists. A tailored form of equations (22)–(25) will then need to be used. In practice, a tailored form of equations (22)–(25) are found through a numerical solver. Simplified boundary element methods or linearized Euler equation solvers can be used to approximate these solutions. This is much like the approach used by Poinsot et al. [54] but represents a major relaxation regarding the propagation of thermoacoustic waves

(iii) Two forms of the Rayleigh criteria are derived analytically. Lord Rayleigh [36] proposed a criterion based upon analysis and observation. Here, the same fluctuations (though in a slightly different form) as $\mathscr{R}_\alpha$ and $\mathscr{R}_\beta$ are derived without assumption from the source terms, Θ_i. They contain the same predictive properties as Strutt's approach, in that they sometimes accurately predict thermoacoustic instability and sometimes do not. This is because they are a reduction of the full two-point cross-correlation of the two-point source terms, $\langle \Theta_i, \Theta_l \rangle$.

Disadvantages are as follows:

(i) Acoustic predictions using equations (37), (38), or (39) or the spectral density equation, (20), result in so-called "infinitesimal" or "linear acoustic" predictions. This is because the solution approach uses the vector Green's function with the concept of the acoustic analogy (see Lighthill [1]). Practically all aeroacoustic prediction methods for flight-vehicles and all methods for external core noise are based on linear acoustics. Currently, there are no known solution approaches for the nonlinear system presented here. However, one possible approach was proposed by the present author that connects a nonlinear propagation solver to source terms (see Miller [58]). Presently, the only way to overcome this shortcoming is to use a fully nonlinear numerical approach such as LES. If one is only concerned with the nature of the source, Θ_i, for thermoacoustic instability or acoustic analysis, then there is no restriction in the present method

(ii) An obvious disadvantage of the present approach is its complexity. However, the presently derived equations used only the most minimal and essential set of assumptions. These assumptions are due to the restriction of no known method to solve this set of partial differential equations analytically.

11. Conclusions

Combustion instability and its prediction remain a difficult engineering challenge since the early 1900s. We have reviewed the canonical theories for the source of combustion noise, which is one of the primary causes of thermoacoustic instability. These source models are often related to wave equation solutions or empirical estimates of instability criteria. We have presented a new theory based on the decomposition of the Navier-Stokes equations coupled with the mass fraction equations. These equations have been decomposed into three major components, and a series of solutions have been presented. Resultant sources and their two-point correlations have been derived. We have identified both combustion-combustion and combustion-aerodynamic source terms, and of these, we identified those that are likely dominant in a low-speed bluff body reacting flow. Using these sources and with some simplifying assumptions, both the classical combustion noise theory and classical Rayleigh criterion have been recovered. However, their recovery shows that their true forms based on the equations of motion are different than originally proposed. Their original form is thus of an empirical nature.

The newly derived closed-form integral equation for the field-variables from reacting flow is consistent with traditionally accepted combustion noise methods. The source terms do not change if a numerical vector Green's function approach is adapted. A spectral prediction method is created, and source terms are consistent with Lord Rayleigh's instability model. The newly proposed Rayleigh criteria models are unique because they involve a volumetric integration of the two-point cross-correlation, which has the benefit of capturing the spatial coherence of the instability unlike the one-point approach. We created models involving two-point cross-correlations to quantify the strength of thermoacoustic radiation from reacting flow. These two-point correlations account for both traditional aeroacoustic sources and combustion noise sources. Particular correlations correspond to the source terms of Lighthill, which are the noise from turbulence. Additional source terms correspond to the unsteady heat release, which is uniquely a combustion-combustion correlation source term.

We hope that the newly outlined analytical theory can be used in practice through new simulations to predict thermoacoustic instability and acoustic radiation. These equations and newly formed theory might inspire specific implementations for the prediction of bluff-body flame holders, rocket engines, or acoustic radiation from high-bypass turbofans.

Nomenclature

Symbols:	Description
c:	Speed of sound
c_p:	Specific heat at constant pressure
$c_{p,k}$:	Specific heat at constant pressure of species k
c_v:	Specific heat at constant volume
$c_{v,k}$:	Specific heat at constant volume of species k
D_k:	Equivalent binary diffusion coefficient of species k
$f_{k,i}$:	Volume force acting on species k
h_s:	Sensible enthalpy
$K_{f,j}$:	Forward rate of reaction j
$K_{r,j}$:	Backward rate of reaction j
M:	Mass within volume
m_k:	Mass of species k within volume
N:	Number of species
p:	Pressure
$\dot{Q}$:	Heat source (spark etc.)
Q_k:	Rate of reaction progress
$\boldsymbol{q}$:	Field-variable vector
q:	Rate of heat release per unit mass
q_i:	Enthalpy flux
$R^{\perp}_{m,n}$:	Two-point space-time cross-correlation of the source terms
R:	Perfect gas constant
r:	Propagation distance of acoustic wave
T:	Temperature
$\mathscr{T}_{\text{comb,aero}}$:	Source terms due to combustion or aerodynamic disturbances
t:	Time
u:	Velocity
$\boldsymbol{V}_{ik}$:	Diffusion velocity of species k
$\boldsymbol{V}_i^c$:	Correction velocity
V:	Volume
W:	Mean molecular weight
W_k:	Molecular weight of species k
X_k:	Mole fraction of species k
$\boldsymbol{x}$:	Spatial coordinate
Y_k:	Mass fraction of species k
$\boldsymbol{y}$:	Source location vector
δ:	Dirac delta function
$\mathscr{F}$:	Flame amplification factor
$\mathscr{R}$:	Rayleigh criteria
Θ_0:	Source vector
$\boldsymbol{\eta}$:	Source vector between $\boldsymbol{x}$ and $\boldsymbol{y}$
τ:	Retarded time
γ:	Ratio of specific heats
ω:	Frequency
$\dot{\omega}_k$:	Mass reaction rate for species k
$\dot{\omega}_T$:	Unsteady heat release due to combustion
ρ:	Density
τ_{ij}:	Viscous stress tensor
λ:	Heat diffusion coefficient
μ:	Dynamic viscosity
ν:	Kinematic viscosity
$\Delta h^o_{f,k}$:	Mass enthalpy of formation of species k and temperature T_o.

Abbreviations

CFD:	Computational fluid dynamics
FWH:	Ffowcs Williams and Hawkings
NSE:	Navier-Stokes equations
RANS:	Reynolds-Averaged Navier-Stokes.

Non-Dimensional Numbers

$\mathscr{L}e_k$:	Lewis number
$\mathscr{P}r$:	Prandtl number
$\mathscr{S}c_k$:	Schmidt number

Data Availability

There are no data associated with this study.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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